Supplementary Material for Disentangling Structural Breaks in Factor Models for Macroeconomic Data

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A Asymptotic Proofs

A.1 Preliminary

First, recall that $V_{NT,1}, V_{NT,2}$ are the $r \times r$ diagonal matrices of the first r largest eigenvalues of the matrices $\frac{1}{T_1 N} X_1 X_1^\top$ and $\frac{1}{T_2 N} X_2 X_2^\top$ respectively. The estimated factor matrices $\tilde{F}_1, \tilde{F}_2$ are $\sqrt{T_1}, \sqrt{T_2}$ times the eigenvectors corresponding to the r largest eigenvalues of $X_1 X_1^\top$ and $X_2 X_2^\top$ respectively. We therefore have for $m = 1, 2$

$$\frac{1}{NT_m} X_m X_m^\top \tilde{F}_m = \tilde{F}_m V_{NT,m}. \tag{A.1}$$

Let $\delta_{NT} = \min\left\{\sqrt{N}, \sqrt{T}\right\}$. Using $X_m = F_m \Lambda_m^\top + e_{(m)}$ gives:

$$\frac{1}{NT_m}\left(F_m \Lambda_m^\top \Lambda_m F_m^\top + F_m \Lambda_m^\top e_{(m)}^\top + e_{(m)} \Lambda_m F_m^\top + e_{(m)} e_{(m)}^\top\right) \tilde{F}_m V_{NT,m}^{-1} = \tilde{F}_m. \tag{A.2}$$

Following Bai (2003) and defining the normalization bases as $H_m = \left(\frac{\Lambda_m^\top \Lambda_m}{N}\right)\left(\frac{F_m^\top \tilde{F}_m}{T_m}\right) V_{NT,m}^{-1}$ for $m = 1, 2$, we have:

$$\tilde{F}_m - F_m H_m = \frac{1}{NT_m}\left(F_m \Lambda_m^\top e_{(m)}^\top \tilde{F}_m + e_{(m)} \Lambda_m F_m^\top \tilde{F}_m + e_{(m)} e_{(m)}^\top\right) V_{NT,m}^{-1}, \tag{A.3}$$

$$\begin{aligned}\tilde{f}_{m,t} - H_m^\top f_t = V_{NT,m}^{-1}\Bigg(&\frac{1}{T_m}\sum_{s=1}^{T} \tilde{f}_{m,s}\gamma_N(s,t)\iota_{mt} + \frac{1}{T_m}\sum_{s=1}^{T} \tilde{f}_{m,s}\zeta_{st}\iota_{mt} \\ &+ \frac{1}{T_m}\sum_{s=1}^{T} \tilde{f}_{m,s}\eta_{m,st}\iota_{mt} + \frac{1}{T_m}\sum_{s=1}^{T} \tilde{f}_s \xi_{m,st}\iota_{mt}\Bigg),\end{aligned} \tag{A.4}$$

where $\zeta_{st} = \frac{e_s^\top e_t}{N} - \gamma_N(s,t), \eta_{m,st} = f_s^\top \Lambda_m^\top e_t / N$ and $\xi_{m,st} = f_t^\top \Lambda_m^\top e_s / N$.

We first present some lemmas from Bai (2003).

Lemma 1. *For $m = 1, 2$:*

(a) Under Assumptions 1 to 4 and 8, $\frac{1}{T_m}\sum_{t=1}^{T}\left\|\left(\tilde{f}_{m,t} - H_m^\top f_t\right)\iota_{mt}\right\|^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

(b) Under Assumptions 1 to 4 and 8, $\frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{m,i} - H_m^{-\top}\lambda_{m,i}\right\|^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

(c) Under Assumptions 1 to 6 and 8, $\frac{1}{T_m}(\tilde{F}_m - F_m H_m)^\top e_{m,i} = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

(d) Under Assumptions 1 to 6 and 8, $\frac{1}{T_m}(\tilde{F}_m - F_m H_m)^\top F_m = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

(e) Under Assumptions 1 to 4 and 8, $\|H_m\| = O_p(1)$.

(f) Under Assumptions 1 to 4, 7 and 8, $\operatorname{plim}\left(\frac{\tilde{F}_m^\top F_m}{T_m}\right) = H_{0,m}^{-1}$, *where we define* V_m *as a diagonal matrix with the eigenvalues of* $\Sigma_{\Lambda_m}^{1/2}\Sigma_{F,m}\Sigma_{\Lambda_m}^{1/2}$, *and* Υ_m *its eigenvectors such that* $\Upsilon_m^\top\Upsilon_m = I_r$, *and* $H_{0,m} = \Sigma_{\Lambda_m}^{1/2}\Upsilon_m V_m^{1/2}$ *for* $m = 1, 2$.

(g) Under Assumptions 1 to 6 and 8, $\frac{1}{T_m\sqrt{N}}(\tilde{F}_m - F_m H_m)^\top \sum_{i=1}^N e_{m,i} = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

Lemmas 1 (a) and 1 (c) to 1 (f) are the subsample counterparts of Lemmas A.1, B.1, B.2, B.3, A.3 and Proposition 1 of Bai (2003) respectively. Lemma 1 (b) follows from Lemma 1 (a) via symmetry. Lemma 1 (g) is nearly identical to Lemma A3 of Han (2015), and can be proven very similarly.

For ease of notation, let $\tilde{F}$ (no subscript) denote the subsample factors joined together *without* any rotation, i.e. $\left[\tilde{F}_1^\top, \tilde{F}_2^\top\right]^\top = \left[\tilde{f}_{1t}, \ldots, \tilde{f}_{1,\lfloor\pi T\rfloor}, \tilde{f}_{2,\lfloor\pi T\rfloor+1}, \ldots, \tilde{f}_{2T}\right]^\top$. Accordingly, let $H_{\iota,t}$ be H_1 or H_2, for $t = 1, \ldots, \lfloor\pi T\rfloor$, for $t = \lfloor\pi T\rfloor + 1, \ldots, T$, respectively.

Lemma 2. *(a) Under Assumptions 1 to 4 and 8,*

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_t - H_{\iota,t}^\top f_t\right)f_t^\top\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)T}\sum_{t=\pi T+1}^{T}\left(\tilde{f}_t - H_{\iota,t}^\top f_t\right)f_t^\top\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right).$$

(b) Under Assumptions 1 to 4 and 8,

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_t - H_{\iota,t}^\top f_t\right)e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)T}\sum_{t=\pi T+1}^{T}\left(\tilde{f}_t - H_{\iota,t}^\top f_t\right)e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right).$$

(c) Under Assumptions 1 to 4 and 8,

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor\sqrt{N}}\sum_{t=1}^{\pi T}\left(\tilde{f}_t-H_{\iota,t}^{\top}f_t\right)\sum_{i=1}^{N}e_{it}\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor(1-\pi)T\rfloor\sqrt{N}}\sum_{t=\pi T+1}^{T}\left(\tilde{f}_t-H_{\iota,t}^{\top}f_t\right)\sum_{i=1}^{N}e_{it}\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right).$$

Proof of Lemma 2 (a).

This is Lemma 3 of Han and Inoue (2015), but with the subsample PC estimates instead, and can therefore be proved very similarly. ■

Proof of Lemma 2 (b).

We shall focus on the first part of this lemma, as the second part can be proven very similarly. First, notice that

$$\begin{aligned}\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_t-H_{\iota,t}^{\top}f_t\right)e_{it}\right\| \leq & \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_t-H_1^{\top}f_t\right)e_{it}\right\| \\ & + \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_{2,t}-H_2^{\top}f_t\right)e_{it}\right\|,\end{aligned}$$

where it suffices to focus on the first term, as the second term can be proven very similarly.

It follows from Equation (A.4) that

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\pi T}\left(\tilde{f}_t-H_1^\top f_t\right)e_{it}\right\| \\
&=\sup_{\pi\in[\pi_1,\pi_2]}\left(V_{NT,1}^{-1}\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)e_{it}+\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\zeta_{s,t}e_{it}\right. \\
&\qquad\left.+\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\eta_{s,t}e_{it}+\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\xi_{s,t}e_{it}\right) \\
&\le\left(\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)e_{it}\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\zeta_{s,t}(s,t)e_{it}\right\|\right. \\
&\qquad\left.+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\eta_{s,t}e_{it}\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2}\sum_{t=1}^{\pi T}\sum_{s=1}^{T}\tilde{f}_{1,s}\xi_{s,t}e_{it}\right\|\right)\left\|V_{NT,1}^{-1}\right\| \\
&=V+VI+VII+VIII.
\end{aligned}
$$

The first term V can be split into

$$
\begin{aligned}
V&=\sup_{\pi\in[\pi_1,\pi_2]}\left(\frac{1}{(\pi T)^2}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\left(\tilde{f}_{1,s}-H_1^\top f_s\right)\gamma_N(s,t)e_{it}+\frac{1}{(\pi T)^2}H_1^\top\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}f_s\gamma_N(s,t)e_{it}\right) \\
&=V.I+V.II, \\
V.I&\le\frac{1}{\pi_1^2}\frac{1}{\sqrt{T}}\left(\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{f}_{1,s}-H_1^\top f_s\right\|^2\right)^{1/2}\sup_{\pi\in[\pi_1,\pi_2]}\left(\frac{1}{T}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}|\gamma_N(s,t)|^2\frac{1}{T}\sum_{t=1}^{\lfloor\pi T\rfloor}e_{it}^2\right)^{1/2} \\
&\le\frac{1}{\pi_1^2}\frac{1}{\sqrt{T}}\left(\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{f}_{1,s}-H_1^\top f_s\right\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_N(s,t)|^2\frac{1}{T}\sum_{t=1}^{T}e_{it}^2\right)^{1/2} \\
&=\frac{1}{\sqrt{T}}O_p\left(\frac{1}{\delta_{NT}}\right), \\
E(V.II)&\le\frac{1}{\pi_1^2}E\left(\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{T^2}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}|\gamma_N(s,t)|\left(\|f_s\|^2\right)^{1/2}\right)\left(Ee_{it}^2\right)^{1/2} \\
&\le\frac{1}{\pi_1^2}E\left(\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_N(s,t)|\left(\|f_s\|^2\right)^{1/2}\right)\left(Ee_{it}^2\right)^{1/2} \\
&\le\frac{M}{T}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_N(s,t)|\right)=O_p\left(\frac{1}{T}\right).
\end{aligned}
$$

The second term VI can be decomposed as

$$
\begin{aligned}
VI &= \sup_{\pi\in[\pi_1,\pi_2]} \left\| \frac{1}{(\pi T)^2} \sum_{t=1}^{\lfloor \pi T \rfloor} \sum_{s=1}^{T} \left(\tilde{f}_{1,s} - H_1^\top f_s \right) \zeta_{s,t} e_{it} + \frac{1}{(\pi T)^2} H_1^\top \sum_{t=1}^{\lfloor \pi T \rfloor} \sum_{s=1}^{T} f_s \zeta_{s,t} e_{it} \right\| \\
&= VI.I + VI.II. \\
VI.I &\leq \frac{1}{\pi_1^2} \left(\frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{f}_{1,s} - H_1^\top f_s \right\|^2 \right)^{1/2} \sup_{\pi\in[\pi_1,\pi_2]} \left(\frac{1}{T} \sum_{s=1}^{T} \left(\frac{1}{T} \sum_{t=1}^{\lfloor \pi T \rfloor} \zeta_{s,t} e_{it} \right)^2 \right)^{1/2}, \\
&= O_p\left(\frac{1}{\delta_{NT}} \right) O_p\left(\frac{1}{\sqrt{N}} \right), \\
VI.II &\leq \frac{1}{\pi_1^2} \sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\sqrt{TN}} \frac{1}{T} \frac{1}{\sqrt{TN}} \sum_{t=1}^{\lfloor \pi T \rfloor} \sum_{s=1}^{T} \sum_{k=1}^{N} f_s \left(e_{ks} e_{kt} - E(e_{ks} e_{kt}) \right) e_{it} \\
&\leq \frac{1}{\pi_1^2} \frac{1}{\sqrt{TN}} \frac{1}{T} \frac{1}{\sqrt{TN}} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{k=1}^{N} f_s \left(e_{ks} e_{kt} - E(e_{ks} e_{kt}) \right) e_{it} \\
&= O_p\left(\frac{1}{\sqrt{TN}} \right),
\end{aligned}
$$

The third term VII can be decomposed as

$$\begin{aligned}
VII &= \sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{(\pi T)^2} \sum_{t=1}^{\lfloor \pi T\rfloor} \sum_{s=1}^{T} \left(\tilde{f}_{1,s} - H_1^\top f_s\right) \eta_{s,t} e_{it} + \frac{1}{(\pi T)^2} \sum_{t=1}^{\lfloor \pi T\rfloor} \sum_{s=1}^{T} H_1^\top f_s \eta_{s,t} e_{it} \\
&= VII.I + VII.II, \\
VII.I &\leq \frac{1}{\pi_1^2} \left(\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{f}_{1,s} - H_1^\top f_s\right\|^2\right)^{1/2} \sup_{\pi\in[\pi_1,\pi_2]} \left(\frac{1}{T}\sum_{s=1}^{T}\left(\frac{1}{T}\sum_{t=1}^{\lfloor \pi T\rfloor}\eta_{st}e_{it}\right)^2\right)^{1/2} \\
&\leq \frac{1}{\pi_1^2} \left(\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{f}_{1,s} - H_1^\top f_s\right\|^2\right)^{1/2} \left(\frac{1}{T}\sum_{s=1}^{T}\left(\frac{1}{T}\sum_{t=1}^{T}\eta_{st}e_{it}\right)^2\right)^{1/2} \\
&= O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\sqrt{N}}\right), \\
VII.II &\leq \frac{1}{\pi_1^2} \sup_{\pi\in[\pi_1,\pi_2]} H_1^\top \left(\frac{1}{T}\sum_{s=1}^{T} f_s f_s^\top\right) \left(\frac{1}{TN}\sum_{t=1}^{\lfloor \pi T\rfloor}\sum_{k=1}^{N}\lambda_k e_{kt} e_{it}\right) \\
&\leq \frac{1}{\pi_1^2} H_1^\top \left(\frac{1}{T}\sum_{s=1}^{T} f_s f_s^\top\right) \left(\frac{1}{TN}\sum_{t=1}^{T}\sum_{k=1}^{N}\lambda_k e_{kt} e_{it}\right) \\
&= O_p\left(\frac{1}{\sqrt{TN}}\right) + O_p\left(\frac{1}{N}\right).
\end{aligned}$$

The negligibility of $VIII$ can be proven in a similar way to VII. Combining the asymptotic orders of V, VI, VII, and $VIII$ proves the lemma. ■

Proof of Lemma 2 (c).

We shall focus on the first part of this lemma, as the second part can be proven very similarly. Similarly to the proof of Lemma 2 (b), notice that

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} \left\| \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\pi T} \left(\tilde{f}_t - H_{\iota,t}^\top f_t\right) \sum_{i=1}^{N} e_{it} \right\| \leq & \sup_{\pi\in[\pi_1,\pi_2]} \left\| \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\pi T} \left(\tilde{f}_t - H_1^\top f_t\right) \sum_{i=1}^{N} e_{it} \right\| \\
& + \sup_{\pi\in[\pi_1,\pi_2]} \left\| \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\pi T} \left(\tilde{f}_{2,t} - H_2^\top f_t\right) \sum_{i=1}^{N} e_{it} \right\|,
\end{aligned}$$

where it suffices to consider the order of the first term. It follows from Equation (A.4) that

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t-H_1^\top f_t\right)\sum_{i=1}^{N}e_{it}\right\| \\
&=\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\pi}\left\|V_{NT,1}^{-1}\left[\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)\sum_{i=1}^{N}e_{it}+\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\zeta_{s,t}\sum_{i=1}^{N}e_{it}\right.\right. \\
&\qquad\left.\left.+\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\eta_{s,t}\sum_{i=1}^{N}e_{it}+\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\epsilon_{s,t}\sum_{i=1}^{N}e_{it}\right]\right\| \\
&\le\left(\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)\sum_{i=1}^{N}e_{it}\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)\sum_{i=1}^{N}e_{it}\right\|\right. \\
&\quad\left.\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)\sum_{i=1}^{N}e_{it}\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(\pi T)^2\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\sum_{s=1}^{T}\tilde{f}_{1,s}\gamma_N(s,t)\sum_{i=1}^{N}e_{it}\right\|\right) \\
&\quad\times\left\|V_{NT,1}^{-1}\right\| \\
&=IX+X+XI+XII.
\end{aligned}
$$

The first two terms are $O_p\left(\frac{1}{\delta_{NT}^2}\right)$, where the specific proof is almost identical to that of Lemma 2 (b), only that Ee_{it}^2 and $\frac{1}{T}\sum_{t=1}^{T}e_{it}^2$ are replaced with $E\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)^2$ and $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)^2$, respectively. The third term XI can be decomposed as

$$
\begin{aligned}
XI&=\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{NT^2}\sum_{s=1}^{T}\sum_{t=1}^{\lfloor\pi T\rfloor}\tilde{f}_{1,s}f_s^\top\Lambda_1e_t\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it} \\
&\le\frac{1}{NT^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{f}_{1,s}f_s^\top\Lambda_1e_t\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it} \\
&=O_p(1)\left(\frac{1}{TN}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\lambda_{1k}e_{kt}e_{it}\right)+\frac{1}{TN\sqrt{N}}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k\neq i}\lambda_{1k}e_{kt}e_{it}\right) \\
&=O_p(1)\left(O_p\left(\frac{1}{N}\right)+O_p\left(\frac{1}{\sqrt{TN}}\right)\right)=O_p\left(\frac{1}{\delta_{NT}\sqrt{N}}\right).
\end{aligned}
$$

The term XII can be decomposed as

$$\begin{aligned}
XII &= \sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{N(\pi T)^2} \sum_{s=1}^{T}\sum_{t=1}^{\lfloor \pi T\rfloor} \tilde{f}_{1,s} e_s^\top \Lambda_1 f_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \\
&= \sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{N(\pi T)^2} \sum_{s=1}^{T}\sum_{t=1}^{T} H_1^\top f_s e_s^\top \Lambda_1 f_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \\
&\quad + \sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{N(\pi T)^2} \sum_{s=1}^{T}\sum_{t=1}^{T} \left(\tilde{f}_{1,s} - H_1^\top f_s\right) e_s^\top \Lambda_1 f_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \\
&= XII.I + XII.II, \\
XII.I &\le \frac{1}{\pi^2}\frac{1}{\sqrt{TN}} \left\| \frac{1}{\sqrt{TN}} \sum_{i=1}^{N}\sum_{s=1}^{T} H_1^\top f_s e_{it} \lambda_{1i}^\top \right\| \left\| \frac{1}{T}\sum_{t=1}^{T} f_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \right\| \\
&\le \frac{1}{\sqrt{TN}} O_p(1) \left(\frac{1}{T}\sum_{t=1}^{T} \|f_t\|^2 \frac{1}{T}\sum_{i=1}^{N} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \right)^2 \right)^{1/2} \\
&= O_p\left(\frac{1}{\sqrt{TN}}\right), \\
XII.II &\le \frac{1}{\sqrt{N}} \left\| \frac{1}{T}\sum_{t=1}^{T} \left(\tilde{f}_t - H_1^\top f_t\right) \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{1i}^\top e_{is} \right) \right\| \left\| \frac{1}{T}\sum_{t=1}^{T} f_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \right\| \\
&\le \frac{1}{\sqrt{N}} \left(\frac{1}{T}\sum_{s=1}^{T} \left\| \tilde{f}_{1,s} - H_1^\top f_s \right\|^2 \frac{1}{T}\sum_{s=1}^{T} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{1i}^\top e_{is} \right\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{T}\sum_{t=1}^{T} \|f_t\|^2 \frac{1}{T} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{it} \right)^2 \right)^{1/2} \\
&= \frac{1}{\sqrt{N}} \left(O_p\left(\frac{1}{\delta_{NT}}\right) O_p(1) \right) O_p(1) = \frac{1}{\sqrt{N}} O_p\left(\frac{1}{\delta_{NT}}\right),
\end{aligned}$$

Combining the asymptotic order of IX, X, XI and XII proves the lemma. ■

A.2 Consistency Proofs

Proposition 1. *Under Assumptions 1 to 8,*

(a) $\left\| \tilde{Z} - H_1^\top Z H_2^{-\top} \right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

(b) $\sup_{\pi\in[\pi_1,\pi_2]} \left\| \tilde{Z} - H_1^\top Z H_2^{-\top} \right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

Proof of Proposition 1.

We begin by proving Proposition 1 (a). By the definition of $\tilde{Z}$, we have:

$$\begin{aligned}\tilde{Z} &= \left(\tilde{\Lambda}_1^\top \tilde{\Lambda}_1\right)^{-1} \tilde{\Lambda}_1^\top \tilde{\Lambda}_2 \\ &= \frac{1}{N} V_{NT,1}^{-1} \frac{1}{T_1} \left(\tilde{F}_1^\top X_1\right)^\top \frac{1}{T_2} \left(\tilde{F}_2^\top X_2\right),\end{aligned}$$

because for $m = 1, 2$, $\tilde{\Lambda}_m^\top \tilde{\Lambda}_m / N = V_{NT,m}$ by eigen-identity, and $\Lambda_m^\top = \left(\tilde{F}_m^\top \tilde{F}\right)^\top \tilde{F}^\top X_m = \frac{1}{T_m} \tilde{F}_m^\top X_m$ via a least squares fit. Therefore

$$\begin{aligned}\tilde{Z} =& V_{NT,1}^{-1} \frac{1}{NT_1T_2} \left(\tilde{F}_1^\top F_1 \Lambda_1^\top + \tilde{F}_1^\top e_1\right) \left(\tilde{F}_2^\top F_2 Z^\top \Lambda_1^\top + \tilde{F}_2^\top F_2 W^\top + \tilde{F}_2^\top e_{(2)}\right)^\top \\ =& V_{NT,1}^{-1} \frac{1}{NT_1T_2} \Big(\tilde{F}_1^\top F_1 \Lambda_1^\top e_{(2)}^\top \tilde{F}_2 + \tilde{F}_1^\top F_1 \Lambda_1^\top W F_2^\top \tilde{F}_2 + \tilde{F}_1^\top F_1 \Lambda_1^\top \Lambda_1 Z F_2^\top \tilde{F}_2 \\ &+ \tilde{F}_1^\top e_{(1)} e_{(2)}^\top \tilde{F}_2 + \tilde{F}_1^\top e_{(1)} W F_2^\top \tilde{F}_2 + \tilde{F}_1^\top e_{(1)} \Lambda_1 Z F_2^\top \tilde{F}_2\Big) \\ =& Z.I + Z.II + Z.III + Z.IV + Z.V + Z.VI.\end{aligned}$$

The first term is negligible because

$$\begin{aligned}Z.I &= \frac{\tilde{F}_1^\top F_1 \Lambda_1^\top e_{(2)}^\top \tilde{F}_2}{NT_1T_2} \\ &= \frac{\tilde{F}_1^\top F_1 \Lambda_1^\top e_{(2)}^\top (\tilde{F}_2 - F_2 H_2)}{T_1 N T_2} + \frac{\tilde{F}_1^\top F_1 \Lambda_1^\top e_{(2)}^\top F_2 H_2}{T_1 N T_2} \\ &\leq \left\| \frac{\tilde{F}_1^\top F_1}{T_1} \right\| \left\| \frac{\Lambda_1^\top e_{(2)}^\top}{N\sqrt{T_2}} \right\| \left\| \frac{\tilde{F}_2 - F_2 H_2}{\sqrt{T_2}} \right\| + \left\| \frac{\tilde{F}_1^\top F_1}{T_1} \right\| \left\| \frac{\Lambda_1^\top e_{(2)}^\top F_2}{NT_2} \right\| \|H_2\| \\ &= O_p(1) O_p\left(\frac{1}{\sqrt{N}}\right) O_p\left(\frac{1}{\delta_{NT}}\right) + O_p(1) O_p\left(\frac{1}{\delta_{NT}^2}\right) O_p(1) \\ &= O_p\left(\frac{1}{\delta_{NT}^2}\right),\end{aligned}$$

because of Lemmas 1 (d) to 1 (f) and Assumptions 1, 3 and 6 respectively.

The second term is negligible because

$$Z.II = \frac{\tilde{F}_1^\top F_1}{T_1} \frac{\Lambda_1^\top W}{N} \frac{F_2^\top \tilde{F}_2}{T_2} = 0,$$

because $\frac{\Lambda_1^\top W}{N} = 0$. Note that we can optionally loosen this condition to $\Lambda_1^\top W/N = O_p\left(\frac{1}{N}\right)$, without affecting the $O_p\left(\frac{1}{\delta_{NT}^2}\right)$ consistency result for $\tilde{Z}$.

The fourth term is negligible because

$$\begin{aligned}
Z.IV &= \frac{\tilde{F}_1^\top e_{(1)} e_{(2)}^\top \tilde{F}_2}{NT_1T_2} \\
&= \frac{(\tilde{F}_1 - F_1H_1)^\top}{T_1} \frac{e_{(1)}e_{(2)}^\top}{N} \frac{(\tilde{F}_2 - F_2H_2)}{T_2} + \frac{(F_1H_1)^\top}{T_1} \frac{e_{(1)}e_{(2)}^\top}{N} \frac{(\tilde{F}_2 - F_2H_2)}{T_2} + \\
&\quad \frac{(\tilde{F}_1 - F_1H_1)^\top}{T_1} \frac{e_{(1)}e_{(2)}^\top}{N} \frac{(F_2H_2)}{T_2} + \frac{(F_1H_1)^\top}{T_1} \frac{e_{(1)}e_{(2)}^\top}{N} \frac{(F_2H_2)}{T_2} \\
&= Z.IV.I + Z.IV.II + Z.IV.III + Z.IV.IV.
\end{aligned}$$

Analysing each of the four terms above, we have:

$$\begin{aligned}
\|Z.IV.I\| &\leq \left\| \frac{(\tilde{F}_1 - F_1H_1)}{\sqrt{T_1}} \right\| \left\| \frac{e_{(1)}e_{(2)}^\top}{\sqrt{T_1}\sqrt{T_2}N} \right\| \left\| \frac{(\tilde{F}_2 - F_2H_2)}{\sqrt{T_2}} \right\| \\
&= O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\delta_{NT}}\right) = O_p\left(\frac{1}{\delta_{NT}^3}\right), \\
\|Z.IV.II\| &\leq \left\| \frac{(F_1H_1)}{\sqrt{T_1}} \right\| \left\| \frac{e_{(1)}e_{(2)}^\top}{\sqrt{T_1}\sqrt{T_2}N} \right\| \left\| \frac{(\tilde{F}_2 - F_2H_2)}{\sqrt{T_2}} \right\| \\
&= O_p(1) O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\delta_{NT}}\right) = O_p\left(\frac{1}{\delta_{NT}^2}\right), \\
\|Z.IV.III\| &\leq \left\| \frac{(\tilde{F}_1 - F_1H_1)}{\sqrt{T_1}} \right\| \left\| \frac{e_{(1)}e_{(2)}^\top}{\sqrt{T_1}\sqrt{T_2}N} \right\| \left\| \frac{(F_2H_2)}{\sqrt{T_2}} \right\| \\
&= O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\delta_{NT}}\right) O_p(1) = O_p\left(\frac{1}{\delta_{NT}^2}\right), \\
\|Z.IV.IV\| &\leq \|H_1\| \left\| \frac{F_1^\top e_{(2)}^\top}{T_1\sqrt{N}} \right\| \left\| \frac{e^\top F_2}{T_2\sqrt{N}} \right\| \|H_2\| \\
&= O_p(1) O_p\left(\frac{1}{\sqrt{T}}\right) O_p\left(\frac{1}{\sqrt{T}}\right) O_p(1) = O_p\left(\frac{1}{\delta_{NT}^2}\right),
\end{aligned}$$

by Lemmas 1 (e) to 1 (f) and Assumption 6. Therefore, $Z.IV = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

The fifth term is negligible because

$$\begin{aligned} Z.V &= \frac{\tilde{F}_1^\top}{T_1}\frac{e_{(1)}W}{N}\frac{F_2^\top \tilde{F}_2}{T_2} \\ &\leq \left\|\frac{(\tilde{F}_1 - F_1 H_1)}{\sqrt{T_1}}\right\|\left\|\frac{e_{(1)}W}{N\sqrt{T_1}}\right\|\left\|\frac{F_2^\top \tilde{F}_2}{T_2}\right\| + \|H\|\left\|\frac{F_1^\top e_{(1)}W}{T_1 N}\right\|\left\|\frac{F_2^\top \tilde{F}_2}{T_2}\right\| \\ &= O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\sqrt{N}}\right) O_p(1) + O_p(1)\, O_p\left(\frac{1}{\delta_{NT}^2}\right) O_p(1) \\ &= O_p\left(\frac{1}{\delta_{NT}^2}\right), \end{aligned}$$

by Lemmas 1 (a), 1 (e) and 1 (f) and Assumption 3. Note that $\left\|\frac{e_{(1)}W}{N\sqrt{T_1}}\right\| = O_p\left(\frac{1}{\sqrt{N}}\right)$ is implied by Assumption 2, because $\|W\| = \|\Lambda_2 - \Lambda_1 Z\| \leq \|\Lambda_2\|$.

The sixth term is negligible because

$$\begin{aligned} Z.VI &= \frac{\tilde{F}_1^\top}{T_1}\frac{e_{(1)}\Lambda_1 Z}{N}\frac{F_2^\top \tilde{F}_2}{T_2} \\ &\leq \left\|\frac{(\tilde{F}_1 - F_1 H_1)}{\sqrt{T_1}}\right\|\left\|\frac{e_{(1)}\Lambda_1}{N\sqrt{T_1}}\right\|\|Z\|\left\|\frac{F_2^\top \tilde{F}_2}{T_2}\right\| + \|H\|\left\|\frac{F_1^\top e_{(1)}\Lambda_1}{T_1 N}\right\|\|Z\|\left\|\frac{F_2^\top \tilde{F}_2}{T_2}\right\| \\ &= O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\sqrt{N}}\right) O_p(1)\, O_p(1) + O_p(1)\, O_p\left(\frac{1}{\delta_{NT}^2}\right) O_p(1)\, O_p(1) \\ &= O_p\left(\frac{1}{\delta_{NT}^2}\right), \end{aligned}$$

by Lemmas 1 (a), 1 (e) and 1 (f) and Assumption 3, and because $\|Z\| < \infty$ is implied by Assumption 2.

Therefore, combining the terms above together, we have:

$$\begin{aligned} \tilde{Z} &= V_{NT}^{-1}\left(\frac{\tilde{F}_1^\top F_1}{T_1}\right)\left(\frac{\Lambda_1^\top \Lambda_1}{N}\right) Z \left(\frac{F_2^\top \tilde{F}_2}{T_2}\right) + O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) \\ &= H_1^\top Z H_2^{-\top} + O_p\left(\frac{1}{\delta_{NT}^2}\right), \end{aligned}$$

where the last line follows from the definition of H_1, and the fact that

$$\begin{aligned} F_2H_2 + \tilde{F}_2 - F_2H_2 &= \tilde{F}_2 \\ \frac{1}{T_2}\tilde{F}_2^\top F_2H_2 + \frac{1}{T_2}\tilde{F}_2^\top\left(\tilde{F}_2 - F_2H_2\right) &= I_r \\ \frac{1}{T_2}\tilde{F}_2^\top F_2H_2 &= I_r + O_p\left(\frac{1}{\delta_{NT}^2}\right) \\ \frac{1}{T_2}F_2^\top\tilde{F}_2 &= H_2^{-\top} + O_p\left(\frac{1}{\delta_{NT}^2}\right). \end{aligned}$$

Rearranging and taking the norms of both sides yields the proof.

For Proposition 1 (b), we have

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{Z} - H_1^\top Z H_2^{-\top}\right\| = \sup_{\pi\in[\pi_1,\pi_2]}\left(Z.I + Z.II + Z.IV + Z.V + Z.VI\right).$$

The supremums of $Z.I$, $Z.II$, $Z.IV$, $Z.V$ and $Z.VI$ can all be proven to be $O_p\left(\frac{1}{\delta_{NT}^2}\right)$ in a very similar way. ■

The consistency results for $\tilde{Z}$ allows us to extend Lemma 1 to the case of the rotated factors in the second subsample.

Lemma 3. *(a) Under Assumptions 1 to 4 and 8,*

$$\begin{aligned} \frac{1}{(1-\pi)T}\left\|\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1)\right\|^2 &= O_p\left(\frac{1}{\delta_{NT}^2}\right), \\ \sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{(1-\pi)T}\left\|\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1)\right\|^2 &= O_p\left(\frac{1}{\delta_{NT}^2}\right). \end{aligned}$$

(b) Under Assumptions 1 to 6 and 8, $\frac{1}{T}\left(\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1\right)^\top e_{(2),i} = O_p\left(\frac{1}{\delta_{NT}^2\sqrt{T}}\right)$,

(c) Under Assumptions 1 to 6 and 8, $\frac{1}{T}\left(\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1\right)^\top F_2 = O_p\left(\frac{1}{\delta_{NT}^2}\right)$.

Proof of Lemma 3.

For Lemma 3 (a), we have

$$\begin{aligned}\frac{1}{\sqrt{T_2}}\left\|\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1\right\| &= \frac{1}{\sqrt{T_2}}\left\|\tilde{F}_2\left(\tilde{Z} - H_1^\top Z H_2^{-\top}\right)^\top + \left(\tilde{F}_2 - F_2H_2\right)\left(H_1^\top Z H_2^{-\top}\right)^\top - F_2Z^\top H_1\right\| \\ &= \left(O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\frac{1}{\delta_{NT}}\right)\right) = O_p\left(\frac{1}{\delta_{NT}}\right),\end{aligned}$$

where the last line follows by Proposition 1 and Lemma 1 (a). Squaring both sides proves the result.

For Lemma 3 (b), we have

$$\begin{aligned}&\frac{1}{T}\left(\tilde{F}_2\tilde{Z}^\top - F_2Z^\top H_1\right)^\top e_{(2),i} \\ =&\frac{1}{T}\left[\tilde{F}_2\left(\tilde{Z}^\top - H_2^{-1}Z^\top H_1\right) + \left(\tilde{F}_2 - F_2H_2\right)H_2^{-1}Z^\top H_1\right]^\top e_{(2),i} \\ =&\frac{1}{T}\left(\tilde{Z}^\top - H_2^{-1}Z^\top H_1\right)^\top \tilde{F}_2^\top e_{(2),i} + \frac{1}{T}H_1^\top Z H_2^{-\top}(\tilde{F}_2 - F_2H_2)^\top e_{(2),i} \\ =&\left(\tilde{Z}^\top - H_2^{-1}Z^\top H_1\right)^\top \left[\frac{(\tilde{F}_2 - F_2H_2)^\top e_{(2),i}}{T} + \frac{F_2^\top e_{(2),i}}{T}\right] + \frac{1}{T}H_1^\top Z H_2^{-\top}(\tilde{F}_2 - F_2H_2)^\top e_{(2),i} \\ =&\left[O_p\left(\frac{1}{\delta_{NT}^2}\right)\right]\left[O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\ =&O_p\left(\frac{1}{\delta_{NT}^2}\right).\end{aligned}$$

For Lemma 3 (c), we have

$$\begin{aligned}&\frac{1}{T}\left(\tilde{F}_2\tilde{Z}^\top F_2Z^\top H_1\right)^\top F_2 \\ =&\frac{1}{T}\left[\tilde{F}_2\left(\tilde{Z}^\top - H_2^{-1}Z^\top H_1\right) + (\tilde{F}_2 - F_2)H_2Z^{-1}Z^\top H_1\right]^\top F_2 \\ =&\frac{1}{T}(\tilde{Z}^\top - H_2^{-1}Z^\top H_1)^\top \tilde{F}_2^\top F_2 + \frac{1}{T}H_1^\top Z H_2^{-\top}\left(\tilde{F}_2 - F_2H_2\right)^\top F_2 \\ =&O_p\left(\frac{1}{\delta_{NT}^2}\right).\end{aligned}$$

■

Next, present some lemmas necessary for the proofs of the Z and W-tests.

Lemma 4. *For $m = 1, 2$:*

(a) under Assumptions 1 to 4 and 8,

$$\frac{1}{T_m}\sum_{t=1}^{T}\left\|\left(\tilde{f}_{m,t} - H_m^\top f_{m,t}\right)\iota_{mt}\right\|^4 = O_p\left(\frac{1}{\delta_{NT}^4}\right),$$
$$\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|\left(\tilde{Z}\tilde{f}_{2,t} - H_1^\top Z f_t\right)\right\|^4 = O_p\left(\frac{1}{\delta_{NT}^4}\right),$$

(b) under Assumptions 1 to 4 and 8, $\frac{1}{T_m}\sum_{t=1}^{T}\left\|\tilde{f}_{m,t}\cdot\iota_{mt}\right\|^4 = O_p(1)$, and $\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|\tilde{Z}\tilde{f}_{2,t}\right\|^4 = O_p(1)$,

(c) under Assumptions 1 to 8, $\|H_m - H_{m,0}\| = O_p\left(\frac{1}{\delta_{NT}}\right)$,

(d) under Assumptions 1 to 4 and 8, $\frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{m,i} - H_m^{-1}\lambda_{m,i}\right\|^4 = O_p\left(\frac{1}{\delta_{NT}^4}\right)$.

Proof of Lemma 4.

Lemmas 4 (a) to 4 (c) are just the subsample counterparts of Lemmas 5.i), 5.ii), 5.iii) and 6 of Han and Inoue (2015), but with the addition of the use of the rotated set of factors.

The second part of Lemma 4 (a) follows by

$$\begin{aligned}
&\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|(\tilde{Z}\tilde{f}_{2,t} - H_1^\top Z f_t)\right\|^4 \\
=&\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|\tilde{Z}\left(\tilde{f}_{2,t} - H_2^\top f_t\right) + \left(\tilde{Z} - H_1^\top Z H_2^{-\top} H_2^\top f_t\right)\right\|^4 \\
\leq&\left\|\tilde{Z}\right\|^4\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|\tilde{f}_{2,t} - H_2^\top f_t\right\|^4 + \left\|\tilde{Z} - H_1^\top Z H_2^{-\top}\right\|\frac{1}{T_2}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left\|H_2^\top f_t\right\|^4 \\
=&O_p\left(\frac{1}{\delta_{NT}^4}\right) + \left[O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\frac{1}{\sqrt{N}}\right)\right]^4 \\
=&O_p\left(\frac{1}{\delta_{NT}^4}\right),
\end{aligned}$$

using Assumption 1, Lemma 1 (a), and Proposition 1.

The second part of Lemma 4 (b) follows by the fact that $\tilde{Z} = O_p(1)$.

Lemma 4 (d) follows by symmetry from Lemma 4 (a).

■

A.3 Z-test Proofs

Define the long run variance estimate of the $A_Z\left(\pi,\widehat{F}\right)$ as $\widehat{S}_Z\left(\pi,\widehat{F}\right)=\frac{1}{\pi}\widehat{\Omega}_{Z,(1)}\left(\pi,\widehat{F}\right)+\frac{1}{1-\pi}\widehat{\Omega}_{Z,(2)}\left(\pi,\widehat{F}\right)$, a weighted average of the variance from pre and post break data ($m=1,2$ respectively):

$$\widehat{\Omega}_{Z,(m)}\left(\pi,\widehat{F}\right)=\widehat{\Gamma}_{(m),0}\left(\pi,\widehat{F}\right)+\sum_{j=1}^{T_m-1}\mathbf{k}\left(\frac{j}{b_{T_m}}\right)\left(\widehat{\Gamma}_{(m),j}\left(\pi,\widehat{F}\right)+\widehat{\Gamma}_{(m),j}\left(\pi,\widehat{F}\right)^{\top}\right), \quad \text{(A.5)}$$

$$\widehat{\Gamma}_{(1),j}\left(\pi,\widehat{F}\right)=\frac{1}{T_1}\sum_{t=j+1}^{T_1}\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^{\top}-I_r\right)\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^{\top}-I_r\right)^{\top}, \quad \text{(A.6)}$$

$$\widehat{\Gamma}_{(2),j}\left(\pi,\widehat{F}\right)=\frac{1}{T_2}\sum_{t=j+T_1+1}^{T}\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^{\top}-I_r\right)\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^{\top}-I_r\right)^{\top}, \quad \text{(A.7)}$$

where $\mathbf{k}(.)$ is a real valued kernel.

The proof of Theorem 3.1 requires proving the following lemmas:

Lemma 5. *(a) Under Assumptions 1 to 8, if $\frac{\sqrt{T}}{N}\to 0$, then*

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|A_Z\left(\pi,\widehat{F}\right)-A_Z(\pi,FH_{0,1})\right\|\xrightarrow{p}0.$$

(b) Under Assumptions 1 to 8, and if the conditions in Assumption 9 hold, then

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)-\widehat{S}_Z(\pi,FH_{0,1})\right\|\xrightarrow{p}0.$$

Proof of Lemma 5 (a).

Taking the supremum of the norm of $A_Z\left(\pi, \widehat{F}\right) - A_Z(\pi, FH_{0,1})$, we have

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]} \left\| A_Z\left(\pi, \widehat{F}\right) - A_Z(\pi, FH_{0,1}) \right\| \\
&= \sup_{\pi\in[\pi_1,\pi_2]} \left\| \text{vech}\sqrt{T}\left(\frac{1}{\pi T}\sum_{t=1}^{\lfloor \pi T\rfloor} \widehat{f}_t\widehat{f}_t^\top - \frac{1}{(1-\pi)T}\sum_{t=\lfloor \pi T+1\rfloor}^{T} \widehat{f}_t\widehat{f}_t^\top \right) \right. \\
&\qquad \left. - \text{vech}\sqrt{T}\left(\frac{1}{\pi T}\sum_{t=1}^{\lfloor \pi T\rfloor} H_{0,1}^\top f_t f_t^\top H_{0,1} - \frac{1}{(1-\pi)T}\sum_{t=\lfloor \pi T+1\rfloor}^{T} H_{0,1}^\top f_t f_t^\top H_{0,1} \right) \right\|.
\end{aligned}$$

Because $\widehat{f}_t$ is consistent for $H_1^\top f_t$, and H_1 is consistent for $H_{0,1}$, it suffices to prove that

$$\begin{aligned}
A_Z.I = \sup_{\pi\in[\pi_1,\pi_2]} \sqrt{T} \left\| \text{vech}\left(\frac{\sum_{t=1}^{\lfloor \pi T\rfloor} \widehat{f}_t\widehat{f}_t^\top}{\pi T} - \frac{\sum_{t=\lfloor \pi T+1\rfloor}^{T} \widehat{f}_t\widehat{f}_t^\top}{(1-\pi)T} \right) - \right. \\
\left. - \text{vech}\left(\frac{\sum_{t=1}^{\lfloor \pi T\rfloor} H_1^\top f_t f_t^\top H_1}{\pi T} - \frac{\sum_{t=\lfloor \pi T+1\rfloor}^{T} H_1^\top f_t f_t^\top H_1}{(1-\pi)T} \right) \right\|,
\end{aligned}$$

and

$$\begin{aligned}
A_Z.II = \sup_{\pi\in[\pi_1,\pi_2]} \sqrt{T} \left\| \text{vech}\left(\frac{\sum_{t=1}^{\lfloor \pi T\rfloor} H_1^\top f_t f_t^\top H_1}{\pi T} - \frac{\sum_{t=\lfloor \pi T+1\rfloor}^{T} H_1^\top f_t f_t^\top H_1}{(1-\pi)T} \right) \right. \\
\left. - \text{vech}\left(\frac{\sum_{t=1}^{\lfloor \pi T\rfloor} H_{0,1}^\top f_t f_t^\top H_{0,1}}{\pi T} - \frac{\sum_{t=\lfloor \pi T+1\rfloor}^{T} H_{0,1}^\top f_t f_t^\top H_{0,1}}{(1-\pi)T} \right) \right\|,
\end{aligned}$$

are both $o_p(1)$. The first term $A_Z.I$ is bounded by

$$\begin{aligned}
A_Z.I \leq \sup_{\pi\in[\pi_1,\pi_2]} \sqrt{T} \left\| \text{vech}\left(\frac{\sum_{t=1}^{\lfloor \pi T\rfloor} \widehat{f}_t\widehat{f}_t^\top - H_1^\top f_t f_t^\top H_1}{\pi T} \right) \right\| \\
+ \sup_{\pi\in[\pi_1,\pi_2]} \sqrt{T} \left\| \text{vech}\left(\frac{\sum_{t=\lfloor \pi T+1\rfloor}^{T} \widehat{f}_t\widehat{f}_t^\top - H_1^\top f_t f_t^\top H_1}{(1-\pi)T} \right) \right\|.
\end{aligned}$$

We focus on proving that the first term is $o_p(1)$, as the second term can be proved very

similarly. The first term is bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left\|\operatorname{vech}\left(\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{f}_t(\widehat{f}_t^\top-H_1^\top f_t)^\top+\widehat{f}_t(H_1^\top f_t)-H_1^\top f_t f_t^\top H_1}{\pi T}\right)\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left\|\operatorname{vech}\left(\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{f}_t(\widehat{f}_t^\top-H_1^\top f_t)+(\widehat{f}_t^\top-f_tH_1)f_t^\top H_1}{\pi T}\right)\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left\|\operatorname{vech}\left(\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}(\widehat{f}_t-H_1^\top f_t)(\widehat{f}_t^\top-f_t^\top H_1)+H_1^\top f_t(\widehat{f}_t^\top-f_t^\top H_1)+(\widehat{f}_t-H_1^\top f_t)f_t^\top H_1}{\pi T}\right)\right\| \\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left(\left\|\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}(\widehat{f}_t-H_1^\top f_t)(\widehat{f}_t^\top-f_t^\top H_1)}{\pi T}\right\|+2\left\|\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}H_1^\top f_t(\widehat{f}_t^\top-f_t^\top H_1)}{\pi T}\right\|\right) \\
=&O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)+O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right) \\
=&o_p(1)
\end{aligned}
$$

as $\frac{\sqrt{T}}{N}\to 0$, where each of the terms on the second last line are $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ by Lemma 1 (a). Next, $A_Z.II$ can be bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left\|\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}H_1^\top f_t f_t^\top H_1-H_{0,1}^\top f_t f_t^\top H_{0,1}}{\pi T}-\frac{\sum_{t=\lfloor\pi T+1\rfloor}^{T}H_1^\top f_t f_t^\top H_1-H_{0,1}^\top f_t f_t^\top H_{0,1}}{(1-\pi)T}\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sqrt{T}\left\|\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}(H_1-H_{0,1})^\top f_t f_t^\top H_1+H_{0,1}^\top f_t f_t^\top(H_1-H_{0,1})}{\pi T}\right. \\
&\qquad\left.-\frac{\sum_{t=\lfloor\pi T+1\rfloor}^{T}(H_1-H_{0,1})^\top f_t f_t^\top H_1+H_{0,1}^\top f_t f_t^\top(H_1-H_{0,1})}{(1-\pi)T}\right\| \\
\leq&\sqrt{T}\|H_1-H_{0,1}\|\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{\sum_{t=1}^{\lfloor\pi T\rfloor}f_t f_t^\top}{\pi T}-\frac{\sum_{t=\lfloor\pi T+1\rfloor}^{T}f_t f_t^\top}{(1-\pi)T}\right\|(\|H_1\|+\|H_{0,1}\|) \\
=&\sqrt{T}o_p(1)O_p\left(\frac{1}{\sqrt{T}}\right)O_p(1),
\end{aligned}
$$

because we use Assumption 8 (b) and $\|H_1-H_{0,1}\|=o_p(1)$ which is implied by Assumptions 1 to 4 and 7 (see Bai (2003)). ■

To prove Lemma 5 (b), we need to prove the following lemmas first.

Lemma 6. *Under Assumptions 1 to 5, for $m = 1, 2$:*

$$\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\widehat{\Gamma}_{(m),j}\left(\pi,\widehat{F}\right)-\widehat{\Gamma}_{(m),j}(\pi,FH_1)\right\|=O_p\left(\frac{1}{\delta_{NT}}\right).$$

Proof of Lemma 6.

We shall focus on the case of $m = 1$, as the case for $m = 2$ is analogous and thus omitted.

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\widehat{\Gamma}_{(1),j}\left(\pi,\widehat{F}\right)-\widehat{\Gamma}_{(1),j}(\pi,FH_1)\right\| \\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^\top-I_r\right)\operatorname{vech}\left(\widehat{f}_{t-j}\widehat{f}_{t-j}^\top-I_r\right)^\top\right. \\
&\left.-\operatorname{vech}\left(H_1^\top f_tf_t^\top H_1-I_r\right)\operatorname{vech}\left(H_1^\top f_{t-j}f_{t-j}^\top H_1-I_r\right)^\top\right\| \\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\{\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^\top-I_r\right)\operatorname{vech}\left(\widehat{f}_{t-j}\widehat{f}_{t-j}^\top-H_1^\top f_{t-j}f_{t-j}^\top H_1\right)\right\|\right. \\
&\left.+\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\operatorname{vech}\left(\widehat{f}_t\widehat{f}_t^\top-H_1^\top f_tf_t^\top H_1\right)\operatorname{vech}\left(H_1^\top f_{t-j}f_{t-j}^\top H_1-I_r\right)^\top\right\|\right\} \\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\{\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\widehat{f}_t^\top\right\|\left\|\widehat{f}_{t-j}\widehat{f}_{t-j}^\top-H_1^\top f_{t-j}f_{t-j}^\top H_1\right\|\right. \\
&+\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}r\left\|\widehat{f}_{t-j}\widehat{f}_{t-j}^\top-H_1^\top f_{t-j}f_{t-j}^\top H_1\right\|+\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\widehat{f}_t^\top-H_1^\top f_tf_t^\top H_1\right\|\left\|H_1^\top f_{t-j}f_{t-j}^\top H_1\right\| \\
&\left.+\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}r\left\|\widehat{f}_j\widehat{f}_j^\top-H_1^\top f_tf_t^\top H_1\right\|\right\} \\
=&\Gamma.I+\Gamma.II+\Gamma.III+\Gamma.IV.
\end{aligned}$$

We proceed by bounding Term $\Gamma.I$:

$$
\begin{aligned}
\Gamma.I = & \sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\widehat{f}_t^{\top}\right\|\left\|\widehat{f}_{t-j}\widehat{f}_{t-j}^{\top}-H_1^{\top}f_{t-j}f_{t-j}^{\top}H_1\right\| \\
\leq & \sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\pi T}\left(\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\right\|^4\right)^{1/2} \\
& \times\left(\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_{t-j}\left(\widehat{f}_{t-j}^{\top}\widehat{f}_{t-j}^{\top}H_1\right)+\left(\widehat{f}_{t-j}-H_1^{\top}f_{t-j}\right)f_{t-j}^{\top}H_1\right\|^2\right)^{1/2} \\
\leq & \sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\right\|^4\right)^{1/2} \\
& \times\left(\frac{2}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_{t-j}\left(\widehat{f}_{t-j}^{\top}-\widehat{f}_{t-j}^{\top}H_1\right)\right\|^2+\frac{2}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\left(\widehat{f}_{t-j}-\widehat{f}_{t-j}H_1\right)f_{t-j}^{\top}H_1\right\|^2\right)^2 \\
\leq & \left(\frac{1}{\pi_1 T}\sum_{t=1}^{T}\left\|\widehat{f}_t\right\|^4\right)^{1/2}\times\left[\left(\frac{2}{\pi_1 T}\sum_{t=1}^{T}\left\|\widehat{f}_t\right\|^4\frac{2}{\pi_1 T}\sum_{t=1}^{T}\left\|\widehat{f}_t-f_tH_1\right\|^4\right)^{1/2}\right. \\
& \left.+\left(\frac{2}{\pi_1 T}\sum_{t=1}^{T}\left\|\widehat{f}_t-H_1^{\top}f_t\right\|^4\frac{2}{\pi_1 T}\sum_{t=1}^{T}\left\|f_tH_1\right\|^4\right)^{1/2}\right]^{1/2} \\
= & O_p\left(\frac{1}{\delta_{NT}}\right),
\end{aligned}
$$

under Assumptions 1 to 5, where $\frac{1}{T}\sum_{t=1}^{T}\left\|\widehat{f}_t\right\|^4 = O_p(1)$ by Lemma 4 (a). Using similar arguments, terms $\Gamma.II$, $\Gamma.III$, and $\Gamma.IV$ can be shown to be $O_p\left(T^{-1/4}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)$. ■

Lemma 7. *Under Assumptions 1 to 5, for $m = 1, 2$,*

$$
\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\widehat{\Gamma}_{m,j}(\pi,F_mH_1)-\widehat{\Gamma}_{m,j}(\pi,F_mH_{0,1})\right\| = O_p\left(\frac{1}{\delta_{NT}}\right).
$$

Proof of Lemma 7.

We shall only prove the lemma for $m = 1$, because the proof for $m = 2$ is similar and thus

omitted.

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\widehat{\Gamma}_{1,j}(\pi,FH_1)-\widehat{\Gamma}_{1,j}(\pi,FH_{0,1})\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left[\operatorname{vech}(H_1^\top f_t f_t^\top H_1-I_r)\operatorname{vech}\left(H_1^\top f_{t-j}f_{t-j}^\top H_1-I_r\right)\right.\right. \\
&\left.\left.-\operatorname{vech}\left(H_{0,1}^\top f_t f_t^\top H_{0,1}-I_r\right)\operatorname{vech}\left(H_{0,1}^\top f_{t-j}f_{t-j}^\top H_{0,1}-I_r\right)^\top\right]\right\| \\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\{\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_t f_t^\top H_1\right\|\left\|H_1^\top f_{t-j}f_{t-j}^\top H_1-H_{0,1}^\top f_{t-j}f_{t-j}^\top H_{0,1}\right\|\right. \\
&+\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_{t-j}f_{t-j}^\top H_1-H_{0,1}^\top f_{t-j}f_{t-j}^\top H_{0,1}\right\|+ \\
&\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\widehat{f}_t\right\|^2\left\|H_1^\top f_t f_t^\top H_1-H_{0,1}^\top f_t f_t^\top H_{0,1}\right\|+ \\
&\left.\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_t f_t^\top H_1-H_{0,1}^\top f_t f_t^\top H_{0,1}\right\|\left\|H_{0,1}^\top f_{t-j}f_{t-j}^\top H_{0,1}\right\|\right\} \\
=&\Gamma.V+\Gamma.VI+\Gamma.VII+\Gamma.VIII.
\end{aligned}
$$

Term $\Gamma.V$ is bounded by:

$$
\begin{aligned}
\Gamma.V=&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_t f_t^\top H_1\right\|\left\|H_1^\top f_{t-j}f_{t-j}^\top H_1-H_{0,1}^\top f_{t-j}f_{t-j}^\top H_{0,1}\right\| \\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\left(\frac{1}{\pi T}\sum_{t=1}^{T}\left\|f_t H_1\right\|^4\right)^{1/2}\left(\frac{1}{\pi T}\left(\|H_1\|^2+\|H_{0,1}\|^2\right)\sum_{t=1}^{T}\|f_t\|^4\right)^{1/2}\|H_1-H_{0,1}\| \\
=&\,O_p(1)\,O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
$$

by Assumption 1 and Lemma 4 (c). The proofs of terms $\Gamma.VI, \Gamma.VII$, and $\Gamma.VIII$ are similar and thus omitted. ■

Proof of Lemma 5 (b).

It suffices to show that

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Omega}_{Z,m}\left(\pi,\widehat{F}\right)-\widehat{\Omega}_{Z,m}\left(\pi,FH_{0,1}\right)\right\|\xrightarrow{p}0,\quad\text{for}\quad m=1,2.$$

For brevity, we will only prove the case for $m=1$, as the case for $m=2$ can be proved similarly. First, see that

$$\begin{aligned}&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Omega}_{Z,1}\left(\pi,\widehat{F}\right)-\widehat{\Omega}_{Z,1}(\pi,FH_{0,1})\right\|\\ \leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Omega}_{Z,1}\left(\pi,\widehat{F}\right)-\widehat{\Omega}_{Z,1}(\pi,FH_1)\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Omega}_{Z,1}\left(\pi,FH_1\right)-\widehat{\Omega}_{Z,1}(\pi,FH_{0,1})\right\|\\ =&\Omega_Z.I+\Omega_Z.II\end{aligned}$$

For the term $\Omega_Z.I$, we have:

$$\begin{aligned}&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Omega}_{Z,1}\left(\pi,\widehat{F}\right)-\widehat{\Omega}_{Z,1}(\pi,FH_1)\right\|\\ \leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Gamma}_{(1),0}\left(\pi,\widehat{F}\right)+\sum_{t=1}^{\lfloor\pi T\rfloor}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\left(\widehat{\Gamma}_{(1),j}\left(\pi,\widehat{F}\right)+\widehat{\Gamma}_{(1),j}\left(\pi,\widehat{F}\right)^\top\right)\right.\\ &\left.-\widehat{\Gamma}_{(1),0}\left(\pi,FH_1\right)-\sum_{t=1}^{\lfloor\pi T\rfloor}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\left(\widehat{\Gamma}_{(1),j}(\pi,FH_1)+\widehat{\Gamma}_{(1),j}(\pi,FH_1)^\top\right)\right\|.\end{aligned}$$

Recall that $\left|\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\right|\leq 1$ and $\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)=0$ if $j>b_{\lfloor\pi T\rfloor}$ for the Bartlett kernel. Thus,

$$\begin{aligned}\Omega_Z.I\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{\Gamma}_{1,0}\left(\pi,\widehat{F}\right)-\widehat{\Gamma}_{1,0}(\pi,FH_1)\right\|+2\sup_{\pi\in[\pi_1,\pi_2]}\sum_{j=1}^{b_{\lfloor\pi T\rfloor}}\left\|\widehat{\Gamma}_{1,j}\left(\pi,\widehat{F}\right)-\widehat{\Gamma}_{1,j}(\pi,FH_1)\right\|\\ =&\,o_p(1),\end{aligned}$$

by Lemma 7 and the condition that $b_T/\delta_{NT}\to 0$ in Assumption 9 (b).

The term $\Omega_Z.II$ can be also be shown to be $o_p(1)$ with similar arguments. ■

To prove the consistency of the Z-test statistic in Theorem 3.1, we present the following lemmas:

Lemma 8. *Under Assumptions 1 to 9,*

(a) $\sup_{\pi\in[\pi_1,\pi_2]} \|A_Z(\pi, FH_1)\| = O_p(1)$ *and* $\sup_{\pi\in[\pi_1,\pi_2]} \left\|A_Z\left(\pi, \widehat{F}\right)\right\| = O_p(1)$,

(b) $\sup_{\pi\in[\pi_1,\pi_2]} \left\|\widehat{S}_Z(\pi, FH_{0,1})^{-1}\right\| = O_p(1)$ *and* $\left\|\widehat{S}_Z\left(\pi, \widehat{F}\right)^{-1}\right\| = O_p(1)$,

(c) $\sup_{\pi\in[\pi_1,\pi_2]} \left\|\widehat{S}_Z\left(\pi, \widehat{F}\right)^{-1} - \widehat{S}_Z(\pi, FH_{0,1})^{-1}\right\| = o_p(1)$.

Proof of Lemma 8. For Lemma 8 (a),

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} A(\pi, FH_1) &= \sup_{\pi\in[\pi_1,\pi_2]} \left\|\text{vech}\left(\frac{1}{\lfloor \pi T\rfloor}\sum_{t=1}^{\pi T} H_1^\top f_t f_t^\top H_1 - \frac{1}{T-\pi T}\sum_{t=\pi T+1}^{T} H_1^\top f_t f_t^\top H_1\right)\right\| \\
&\leq \sup_{\pi\in[\pi_1,\pi_2]} \left\|\frac{1}{\lfloor \pi T\rfloor}\sum_{t=1}^{\pi T} f_t f_t^\top - \frac{1}{T-\pi T}\sum_{t=\pi T+1}^{T} f_t f_t^\top\right\| \|H_1\|^2 \\
&= O_p(1),
\end{aligned}$$

by Assumption 8 (b), and

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} \left\|A_Z\left(\pi, \widehat{F}\right)\right\| &\leq \sup_{\pi\in[\pi_1,\pi_2]} \|A_Z(\pi, FH_{0,1})\| + \sup_{\pi\in[\pi_1,\pi_2]} \left\|A_Z\left(\pi, \widehat{F}\right) - A_Z(\pi, FH_{0,1})\right\| \\
&= O_p(1) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),
\end{aligned}$$

by Lemma 5 (a).

For Lemma 8 (b), because $0 < \pi < 1$, this implies that $\sup_{\pi\in[\pi_1,\pi_2]} \left\|\widehat{S}_Z(\pi, FH_{0,1}) - \left(\left(\frac{1}{\pi} + \frac{1}{1-\pi}\right)\Omega_Z\right)\right\| = o_p(1)$. Since Ω_Z is positive definite, $\sup_{\pi\in[\pi_1,\pi_2]} \left|\rho_{min}\left(\widehat{S}_Z(\pi, FH_{0,1})\right) - \rho_{min}\left(\left(\frac{1}{\pi} + \frac{1}{1-\pi}\right)\Omega_Z\right)\right| \leq \sup_{\pi\in[\pi_1,\pi_2]} \left\|\widehat{S}_Z(\pi, FH_{0,1}) - \left(\frac{1}{\pi} + \frac{1}{1-\pi}\right)\Omega_Z\right\| = o_p(1)$, where ρ_{min}, ρ_{max} denote the minimum and maximum eigenvalues of a symmetric matrix, respectively. Thus, the eigenvalues of $\widehat{S}_Z(\pi, FH_{0,1})$ are bounded away from zero in π,

and $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z(\pi, FH_{0,1})\right\| = O_p(1)$. For the second part of this lemma, we have

$$\begin{aligned}\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)-\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_Z\right\| \leq & \sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)-\widehat{S}_Z(\pi,FH_{0,1})\right\| \\ & + \sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z(\pi,FH_{0,1})-\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_Z\right\| \\ = & \, o_p(1),\end{aligned}$$

by Lemma 5 (b) and Assumption 9 (a). Therefore,

$$\begin{aligned}&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\rho_{min}\left(\widehat{S}_Z\left(\pi,\widehat{F}\right)\right)-\rho_{min}\left(\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_Z\right)\right\| \\ \leq & \sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)-\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_Z\right\| = o_p(1),\end{aligned}$$

which means that the eigenvalues of $\widehat{S}_Z\left(\pi,\widehat{F}\right)$ are also bounded away from zero, implying that $\widehat{S}_Z\left(\pi,\widehat{F}\right)^{-1} = O_p(1)$.

For Lemma 8 (c):

$$\begin{aligned}&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)^{-1}-\widehat{S}_Z(\pi,FH_{0,1})^{-1}\right\| \\ =&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z(\pi,FH_{0,1})^{-1}\left(\widehat{S}_Z(\pi,FH_{0,1})-\widehat{S}_Z\left(\pi,\widehat{F}\right)\right)\widehat{S}_Z\left(\pi,\widehat{F}\right)^{-1}\right\| \\ \leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z(\pi,FH_{0,1})^{-1}\right\|\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z(\pi,FH_{0,1})-\widehat{S}_Z\left(\pi,\widehat{F}\right)\right\|\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{S}_Z\left(\pi,\widehat{F}\right)^{-1}\right\| \\ =&O_p(1)\,o_p(1)O_p(1) = o_p(1),\end{aligned}$$

by Lemma 5 (b) and Assumption 9 (a). ■

Proof of Theorem 3.1.

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left|\mathscr{W}_Z\left(\pi,\widehat{F}\right)-\mathscr{W}_Z(\pi,FH_{0,1})\right| \\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left|A_Z\left(\pi,\widehat{F}\right)^\top\left[\widehat{S}_Z\left(\pi,\widehat{F}\right)^{-1}-\widehat{S}_Z(\pi,FH_{0,1})^{-1}\right]A_Z\left(\pi,\widehat{F}\right)\right| \\
&+\sup_{\pi\in[\pi_1,\pi_2]}\left|\left[A_Z\left(\pi,\widehat{F}\right)-A_Z(\pi,FH_{0,1})\right]^\top\widehat{S}_Z(\pi,FH_{0,1})^{-1}A_Z\left(\pi,\widehat{F}\right)\right| \\
&+\sup_{\pi\in[\pi_1,\pi_2]}\left|A_Z(\pi,FH_{0,1})^\top\widehat{S}_Z(\pi,FH_{0,1})^{-1}\left[A_Z\left(\pi,\widehat{F}\right)-A_Z(\pi,FH_{0,1})\right]\right| \\
=&o_p(1),
\end{aligned}
$$

using the results of Lemma 8 and Lemma 5 (a). ■

Proof of Theorem 3.2.

Under the alternative hypothesis for the Z-test, we have:

$$
\begin{aligned}
&\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{f}_t\widehat{f}_t^\top-\frac{1}{T-\lfloor\pi T\rfloor}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\widehat{f}_t\widehat{f}_t^\top \\
=&\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}H_1^\top f_tf_t^\top H_1-\frac{1}{T-\lfloor\pi T\rfloor}\sum_{t=\lfloor\pi T+1\rfloor}^{T}H_1^\top Zf_tf_t^\top Z^\top H_1\right) \\
&+\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\widehat{f}_t\widehat{f}_t^\top-H_1^\top f_tf_t^\top H_1\right)-\frac{1}{T-\lfloor\pi T\rfloor}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\left(\widehat{f}_t\widehat{f}_t^\top-H_1^\top Zf_tf_t^\top Z^\top H_1\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\widehat{f}_t\widehat{f}_t^\top-H_1^\top f_tf_t^\top H_1\right) \\
=&\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left[\left(\widehat{f}_t-H_1^\top f_t\right)f_t^\top H_1+\left(\widehat{f}_t-H_1^\top f_t\right)\left(\widehat{f}_t^\top-f_t^\top H_1\right)+H_1^\top f_t\left(\widehat{f}_t^\top-f_t^\top H_1\right)\right] \\
=&O_p\left(\frac{1}{\delta_{NT}^2}\right),
\end{aligned}
$$

by the arguments in the proof of Lemma 5 (a). Similarly, $\frac{1}{T-\lfloor\pi T\rfloor}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\left(\widehat{f}_t\widehat{f}_t^\top-H_1^\top Zf_tf_t^\top Z^\top H_1\right)=$

$O_p\left(\frac{1}{\delta_{NT}^2}\right)$. The main dominating term is thus

$$H_1^\top \left(\frac{1}{\lfloor \pi T \rfloor} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t f_t^\top - \frac{1}{T - \lfloor \pi T \rfloor} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} Z f_t f_t^\top Z^\top \right) H_1 \xrightarrow{p} H_{0,1}^\top \left(\Sigma_F - Z \Sigma_F Z^\top\right) H_{0,1} \equiv C,$$

by Assumption 8, and the definitions of H_1 and $H_{0,1}$. Matrix C contains non-zero entries because $\Sigma_F - Z\Sigma_F Z^\top$ is not zero by Assumption Z2, and the fact that $H_{0,1}$ is non-singular. Note that Assumptions 1 to 8 still hold under the alternative hypothesis albeit with an appropriate pseudo factor representation, and hence Lemma 5 (b) still hold for the equivalent models under the alternative. Finally, putting the above together we have:

$$\begin{aligned}
&\mathscr{W}_Z\left(\pi, \widehat{F}\right) \\
=&A_Z\left(\pi, \widehat{F}\right)^\top \widehat{S}_Z\left(\pi, \widehat{F}\right)^{-1} A_Z\left(\pi, \widehat{F}\right) \\
=&\frac{T}{\max\left(b_{\lfloor \pi T \rfloor}, b_{T-\lfloor \pi T \rfloor}\right)} \left[\frac{1}{\sqrt{T}} A_Z\left(\pi, \widehat{F}\right)^\top\right] \left[\max\left(b_{\lfloor \pi T \rfloor}, b_{T-\lfloor \pi T \rfloor}\right) \widehat{S}_Z\left(\pi, \widehat{F}\right)^{-1}\right] \left[\frac{1}{\sqrt{T}} A_Z\left(\pi, \widehat{F}\right)\right] \\
=&\frac{T}{\max\left(b_{\lfloor \pi T \rfloor}, b_{T-\lfloor \pi T \rfloor}\right)} \left[\mathrm{vech}(C)^\top + o_p(1)\right] \left[\max\left(b_{\lfloor \pi T \rfloor}, b_{T-\lfloor \pi T \rfloor}\right) \widehat{S}_Z\left(\pi, \widehat{F}\right)^{-1}\right] \left[\mathrm{vech}(C) + o_p(1)\right] \\
\to&\infty,
\end{aligned}$$

by Assumption Z2. The consistency of $\sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_Z\left(\pi, \widehat{F}\right)$ follows from the consistency of $\mathscr{W}_Z\left(\pi, \widehat{F}\right)$. ■

A.4 W-test Proofs

We first recall the following identity for the factor loadings for $m = 1, 2$,

$$\tilde{\lambda}_{m,i} - H_m^{-1} \lambda_{m,i} = \frac{1}{T_m} H_m^\top F_m^\top e_{(m),i} + \frac{1}{T_m} \tilde{F}^\top \left(F_m - \tilde{F}_m H_m^{-1}\right) \lambda_{m,i} + \frac{1}{T_m} \left(\tilde{F}_m - F_m H_m\right)^\top e_{(m),i}, \tag{A.8}$$

where Equation (A.8) is the subsample version of the asymptotic expansion of the factor loadings considered by Bai (2003) (see the proof of their Theorem 2).

Next, recalling that $\tilde{w}_i = \tilde{\lambda}_{2i} - \tilde{Z}^\top \tilde{\lambda}_{1i}$, we derive the following identity for $\tilde{w}_i$:

$$\begin{aligned}
\tilde{w}_i &= H_2^{-1}\lambda_{2i} + H_2^\top \frac{1}{\lfloor (1-\pi)T \rfloor} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} + \frac{1}{\lfloor (1-\pi)T \rfloor} \tilde{F}_2^\top \left(F_2 - \tilde{F}_2 H_2^{-1} \right) \lambda_{2i} \\
&\quad + \frac{1}{\lfloor (1-\pi)T \rfloor} \left(\tilde{F}_2 - F_2 H_2 \right)^\top e_{(2),i} - \tilde{Z}^\top H_1^{-1} \lambda_{1i} \\
&\quad - \tilde{Z}^\top \left[H_1^\top \frac{1}{\lfloor (1-\pi)T \rfloor} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} + \frac{1}{\lfloor \pi T \rfloor} \tilde{F}_1^\top \left(F_1 - \tilde{F}_1 H_1^{-1} \right) \lambda_{1i} + \frac{1}{\lfloor \pi T \rfloor} \left(\tilde{F}_1 - F_1 H_1 \right)^\top e_{(1),i} \right] \\
&= H_2^{-1}\lambda_{2i} - H_2^{-1} Z^\top \lambda_{1i} + \left[\tilde{Z}^\top - H_2^{-1} Z^\top H_1 \right] H_1^{-1} \lambda_{1i} \\
&\quad + H_2^\top \frac{1}{\lfloor (1-\pi)T \rfloor} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} + \frac{1}{\lfloor (1-\pi)T \rfloor} \tilde{F}_2^\top \left(F_2 - \tilde{F}_2 H_2^{-1} \right) \lambda_{2i} \\
&\quad - \tilde{Z}^\top \left[H_1^\top \frac{1}{\lfloor (1-\pi)T \rfloor} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} + \frac{1}{\lfloor \pi T \rfloor} \tilde{F}_1^\top \left(F_1 - \tilde{F}_1 H_1^{-1} \right) \lambda_{1i} + \frac{1}{\lfloor \pi T \rfloor} \left(\tilde{F}_1 - F_1 H_1 \right)^\top e_{(1),i} \right]
\end{aligned}$$

Rearranging the terms then yields

$$\begin{aligned}
\tilde{w}_i - H_2^{-1} w_i &= H_2^\top \frac{1}{\lfloor (1-\pi)T \rfloor} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} + \frac{1}{\lfloor (1-\pi)T \rfloor} \tilde{F}_2^\top \left(F_2 - \tilde{F}_2 H_2^{-1} \right) \lambda_{2i} \\
&\quad + \frac{1}{\lfloor (1-\pi)T \rfloor} \left(\tilde{F}_2 - F_2 H_2 \right)^\top e_{(2),i} \\
&\quad - \tilde{Z}^\top \left[H_1^\top \frac{1}{\lfloor \pi T \rfloor} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} + \frac{1}{\lfloor \pi T \rfloor} \tilde{F}_1^\top \left(F_1 - \tilde{F}_1 H_1^{-1} \right) \lambda_{1i} \right. \\
&\qquad \left. + \frac{1}{\lfloor \pi T \rfloor} \left(\tilde{F}_1 - F_1 H_1 \right)^\top e_{(1),i} \right] \\
&\quad - \left[\tilde{Z}^\top - H_2^{-1} Z^\top H_1 \right] H_1^{-1} \lambda_{1i}. \qquad \text{(A.9)}
\end{aligned}$$

Lemma 9. *Under Assumptions 1 to 8, as* $\sqrt{T}/N \to 0$,

(a)

$$\begin{aligned}\sqrt{T}\left(\tilde{w}_i - H_2^{-1}w_i\right) &= H_2^\top \frac{1}{(1-\pi)\sqrt{T}} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} \\ &\quad - H_2^{-1} Z^\top H_1 H_1^\top \frac{1}{\pi\sqrt{T}} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} + o_p(1), \\ \sup_{\pi\in[\pi_1,\pi_2]} \left\{\sqrt{T}\left(\tilde{w}_i - H_2^{-1}w_i\right)\right\} &= \sup_{\pi\in[\pi_1,\pi_2]} \left\{ H_2^\top \frac{1}{(1-\pi)\sqrt{T}} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} \right. \\ &\quad \left. - H_2^{-1} Z^\top H_1 H_1^\top \frac{1}{\pi\sqrt{T}} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} \right\} + o_p(1).\end{aligned}$$

(b) Additionally, if Assumptions W1 to W3 hold,

$$\begin{aligned}\sqrt{TN}\frac{\sum_{i=1}^N \tilde{w}_i - H_2^{-1}w_i}{N} &= H_2^\top \frac{1}{(1-\pi)\sqrt{TN}} \sum_{i=1}^{N} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} \\ &\quad - H_2^{-1} Z^\top H_1 H_1^\top \frac{1}{\pi\sqrt{TN}} \sum_{i=1}^{N} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} + o_p(1), \\ \sup_{\pi\in[\pi_1,\pi_2]} \left\{\sqrt{TN}\frac{\sum_{i=1}^N \tilde{w}_i - H_2^{-1}w_i}{N}\right\} &= \sup_{\pi\in[\pi_1,\pi_2]} \left\{ H_2^\top \frac{1}{(1-\pi)\sqrt{TN}} \sum_{i=1}^{N} \sum_{t=\lfloor \pi T+1 \rfloor}^{T} f_t e_{it} \right. \\ &\quad \left. - H_2^{-1} Z^\top H_1 H_1^\top \frac{1}{\pi\sqrt{TN}} \sum_{i=1}^{N} \sum_{t=1}^{\lfloor \pi T \rfloor} f_t e_{it} \right\} + o_p(1).\end{aligned}$$

Proof of Lemma 9.

For Lemma 9 (a), using Equation (A.9), we have

$$\begin{aligned}
\sqrt{T}\left(\tilde{w}_i - H_2^{-1}w_i\right) &= H_2^{\top}\frac{1}{(1-\pi)\sqrt{T}}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} + \frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2 H_2^{-1}\right)\lambda_{2i} \\
&\quad + \frac{1}{(1-\pi)\sqrt{T}}\left(\tilde{F}_2 - F_2 H_2\right)^{\top} e_{(2),i} \\
&\quad - \tilde{Z}^{\top}\left[H_2^{\top}\frac{1}{\pi\sqrt{T}}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it} + \frac{1}{\pi\sqrt{T}}\tilde{F}_2^{\top}\left(F_1 - \tilde{F}_1 H_1^{-1}\right)\lambda_{1i}\right. \\
&\qquad \left. + \frac{1}{\pi\sqrt{T}}\left(\tilde{F}_1 - F_1 H_1\right)^{\top} e_{(1),i}\right] - \sqrt{T}\left(\tilde{Z}^{\top} - H_2^{-1}Z^{\top}H_1\right)H_1^{-1}\lambda_{1i} \\
&= H_2^{\top}\frac{1}{(1-\pi)\sqrt{T}}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} - \tilde{Z}^{\top}H_1^{\top}\frac{1}{\pi\sqrt{T}}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it} \\
&\quad + \frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2 H_2^{-1}\right)\lambda_{2i} + \frac{1}{(1-\pi)\sqrt{T}}\left(\tilde{F}_2 - F_2 H_2\right)^{\top} e_{(2),i} \\
&\quad - \tilde{Z}^{\top}\frac{1}{\pi\sqrt{T}}\tilde{F}_1^{\top}\left(F_1 - \tilde{F}_1 H_1^{-1}\right)\lambda_{1i} - \tilde{Z}^{\top}\frac{1}{\pi\sqrt{T}}\left(\tilde{F}_1 - F_1 H_1\right)^{\top} e_{(1),i} \\
&\quad - \sqrt{T}\left(\tilde{Z}^{\top} - H_2^{-1}Z^{\top}H_1\right)H_1^{-1}\lambda_{1i} \\
&= W.I + W.II + W.III + W.IV + W.V + W.VI + W.VII,
\end{aligned}$$

where $W.III$ and $W.V$ are $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Lemma 1 (d), $W.IV$ and $W.VI$ are $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Lemma 1 (c), and $W.VII$ is $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Proposition 1 (a). Combining the asymptotic order of these terms yields the required result.

For the second part of this lemma, we have

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\{\sqrt{T}\left(\tilde{w}_i - H_2^{-1}w_i\right)\right\} \\
&= \sup_{\pi\in[\pi_1,\pi_2]}\left\{W.I + W.II + W.III + W.IV + W.V + W.VI + W.VII\right\}, \\
&= \sup_{\pi\in[\pi_1,\pi_2]}\left\{W.I + W.II\right\} + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),
\end{aligned}$$

because

$$\sup_{\pi\in[\pi_1,\pi_2]}\|W.III\| = \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^\top\left(F_2-\tilde{F}_2H_2^{-1}\right)\lambda_{2i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\|W.IV\| = \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)\sqrt{T}}\left(\tilde{F}_2-F_2H_2\right)^\top e_{(2),i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\|W.V\| = \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{Z}^\top\frac{1}{\pi\sqrt{T}}\tilde{F}_1^\top\left(F_1-\tilde{F}_1H_1^{-1}\right)\lambda_{1i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\|W.VI\| = \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{Z}^\top\frac{1}{\pi\sqrt{T}}\left(\tilde{F}_1-F_1H_1\right)^\top e_{(1),i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\|W.VII\| = \sup_{\pi\in[\pi_1,\pi_2]}\left\|\sqrt{T}\left(\tilde{Z}^\top-H_2^{-1}Z^\top H_1\right)H_1^{-1}\lambda_{1i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),$$

due to Lemmas 2 (a) and 2 (b) and Proposition 1 (b). Combining the asymptotic order of these terms yields the required result.

For Lemma 9 (b), using Equation (A.9) and taking the cross-sectional average of both

sides, we have

$$
\begin{aligned}
&\sqrt{TN}\frac{\sum_{i=1}^{N}\left(\tilde{w}_i - H_2^{-1}w_i\right)}{N} \\
&= H_2^{\top}\frac{1}{(1-\pi)\sqrt{TN}}\sum_{i=1}^{N}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} + \frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2 H_2^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{2i} \\
&\quad + \frac{1}{(1-\pi)\sqrt{TN}}\left(\tilde{F}_2 - F_2 H_2\right)^{\top}\sum_{i=1}^{N} e_{(2),i} \\
&\quad - \tilde{Z}^{\top}\left[H_2^{\top}\frac{1}{\pi\sqrt{TN}}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it} + \frac{1}{\pi\sqrt{T}}\tilde{F}_2^{\top}\left(F_1 - \tilde{F}_1 H_1^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i}\right. \\
&\qquad \left. + \frac{1}{\pi\sqrt{TN}}\left(\tilde{F}_1 - F_1 H_1\right)^{\top}\sum_{i=1}^{N} e_{(1),i}\right] \\
&\quad - \sqrt{T}\left(\tilde{Z}^{\top} - H_2^{-1}Z^{\top}H_1\right)H_1^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i} \\
&= H_2^{\top}\frac{1}{(1-\pi)\sqrt{TN}}\sum_{i=1}^{N}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} - \tilde{Z}^{\top}H_1^{\top}\frac{1}{\pi\sqrt{TN}}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it} \\
&\quad + \frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2 H_2^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{2i} + \frac{1}{(1-\pi)\sqrt{TN}}\left(\tilde{F}_2 - F_2 H_2\right)^{\top}\sum_{i=1}^{N} e_{(2),i} \\
&\quad - \tilde{Z}^{\top}\frac{1}{\pi\sqrt{T}}\tilde{F}_1^{\top}\left(F_1 - \tilde{F}_1 H_1^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i} - \tilde{Z}^{\top}\frac{1}{\pi\sqrt{TN}}\left(\tilde{F}_1 - F_1 H_1\right)^{\top}\sum_{i=1}^{N} e_{(1),i} \\
&\quad - \sqrt{T}\left(\tilde{Z}^{\top} - H_2^{-1}Z^{\top}H_1\right)H_1^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i} \\
&= W.VIII + W.IX + W.X + W.XI + W.XII + W.XIII + W.XIV,
\end{aligned}
$$

where $W.X$ and $W.XII$ are $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Lemma 1 (d), $W.XI$ and $W.XIII$ are $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Lemma 1 (g), and $W.XIV$ is $O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ due to Proposition 1 (a). Combining the asymptotic order of these terms yields the required result.

For the second part of this lemma, we have

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\{\sqrt{TN}\frac{\sum_{i=1}^{N}\left(\tilde{w}_i - H_2^{-1}w_i\right)}{N}\right\} \\
&= \sup_{\pi\in[\pi_1,\pi_2]}\left\{W.VIII + W.IX + W.X + W.XI + W.XII + W.XIII + W.IV\right\}, \\
&= \sup_{\pi\in[\pi_1,\pi_2]}\left\{W.VIII + W.IX\right\} + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),
\end{aligned}$$

because

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\|W.X\| &= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2H_2^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{2i}\right\| \\
&= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)\sqrt{T}}\tilde{F}_2^{\top}\left(F_2 - \tilde{F}_2H_2^{-1}\right)\right\|\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{2i} = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),\right. \\
\sup_{\pi\in[\pi_1,\pi_2]}\|W.XI\| &= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{(1-\pi)\sqrt{TN}}\left(\tilde{F}_2 - F_2H_2\right)^{\top}\sum_{i=1}^{N}e_{(2),i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right), \\
\sup_{\pi\in[\pi_1,\pi_2]}\|W.XII\| &= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{Z}^{\top}\frac{1}{\pi\sqrt{T}}\tilde{F}_1^{\top}\left(F_1 - \tilde{F}_1H_1^{-1}\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right), \\
\sup_{\pi\in[\pi_1,\pi_2]}\|W.XIII\| &= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{Z}^{\top}\frac{1}{\pi\sqrt{TN}}\left(\tilde{F}_1 - F_1H_1\right)^{\top}\sum_{i=1}^{N}e_{(1),i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right), \\
\sup_{\pi\in[\pi_1,\pi_2]}\|W.XIV\| &= \sup_{\pi\in[\pi_1,\pi_2]}\left\|\sqrt{T}\left(\tilde{Z}^{\top} - H_2^{-1}Z^{\top}H_1\right)H_1^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{1i}\right\| = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right),
\end{aligned}$$

where the result for $\sup_{\pi\in[\pi_1,\pi_2]}\|W.X\|$ follows by Assumption W1 and Lemma 2 (a), $\sup_{\pi\in[\pi_1,\pi_2]}\|W.XI\|$ and $\sup_{\pi\in[\pi_1,\pi_2]}\|W.XIII\|$ follow by Lemma 2 (a), and $\sup_{\pi\in[\pi_1,\pi_2]}\|W.XII\|$ and $\sup_{\pi\in[\pi_1,\pi_2]}\|W.XIV\|$ follow by Lemma 2 (c), Proposition 1 (b), and Lemma 2 (c). ■

For ease of notation, define $\widehat{e}_{it} = \tilde{e}_{(m),it}\cdot\iota_{mt}$ for $m = 1, 2$.

Lemma 10. *Under Assumptions 1 to 6 and 8 and Assumptions W2 and W3, and if* $\sqrt{T}/N \to 0$*, then for* $m = 1, 2$*:*

(a)

$$\frac{1}{T_m}\sum_{t=1}^{T}\left|\left(\tilde{e}_{(m),it}-e_{it}\right)\cdot\iota_{mt}\right|^4=o_p(1),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi T}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{it}-e_{it}\right|^4\right\}=o_p(1),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{(1-\pi)T}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{it}-e_{it}\right|^4\right\}=o_p(1),$$

(b) If Assumption W1 additionally holds, then

$$\frac{1}{T_mN}\sum_{i=1}^{N}\sum_{t=1}^{T}\left|\left(\tilde{e}_{(m),it}-e_{it}\right)\cdot\iota_{mt}\right|^4=o_p(1),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi TN}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{it}-e_{it}\right|^4\right\}=o_p(1),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{(1-\pi)TN}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{it}-e_{it}\right|^4\right\}=o_p(1),$$

(c)

$$\frac{1}{T_m}\sum_{t=1}^{T}\left(\tilde{e}_{(m),it}\iota_{mt}\right)^4=O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi T}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{e}_{it}^4\right\}=O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{(1-\pi)T}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{e}_{it}^4\right\}=O_p\left(1\right),$$

(d) If Assumption W1 additionally holds,

$$\frac{1}{T_m N}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\tilde{e}_{(m),it}\cdot\iota_{mt}\right)^4 = O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi TN}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{e}_{it}^4\right\} = O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{(1-\pi T)N}\sum_{i=1}^{N}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\widehat{e}_{it}^4\right\} = O_p\left(1\right).$$

Proof of Lemma 10 (a).

For brevity, we focus on $m=1$, as the case for $m=2$ is very similar. We have by definition:

$$\begin{aligned}
\left|\tilde{e}_{(1),it}-e_{it}\right| &= \left|\tilde{\lambda}_{1,i}^\top\tilde{f}_t-\lambda_{1,i}^\top f_t\right| \\
&= \left|\tilde{\lambda}_{1,i}^\top\tilde{f}_t-\lambda_{1,i}^\top H_1^{-\top}H_1^\top f_t\right| \\
&= \left|\tilde{\lambda}_{1,i}^\top\left(\tilde{f}_t-H_1^\top f_t\right)+\tilde{\lambda}_{1,i}^\top H_1^\top f_t-\left(\lambda_{1,i}^\top H_1^{-\top}H_1^\top f_t\right)\right| \\
&= \left|\tilde{\lambda}_{1,i}^\top\left(\tilde{f}_t-H_1^\top f_t\right)+\left(\tilde{\lambda}_{1,i}^\top-\lambda_{1,i}^\top H_1^{-\top}\right)H_1^\top f_t\right| \\
&= |E.I_t+E.II_t|.
\end{aligned}$$

Noting that $\left|\tilde{e}_{(1),it}-e_{it}\right|^4 \leq 64|E.I_t^4+E.II_t^4|$, it therefore suffices to consider the bounds of $\frac{1}{T_1}\sum_{t=1}^{T}E.I_t^4$ and $\frac{1}{T_1}\sum_{t=1}^{T}E.II_t^4$.

First, $\frac{1}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}E.I_t^4$ can be bounded by:

$$\begin{aligned}
\frac{1}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}E.I_t^4 &\leq \left\|\tilde{\lambda}_{1,i}\right\|^4\frac{1}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t-H_1^\top f_t\right\|^4 \\
&= O_p\left(1\right)O_p\left(\frac{1}{\delta_{NT}^4}\right),
\end{aligned}$$

where $\left\|\tilde{\lambda}_{1,i}\right\|^4 = O_p\left(1\right)$ after suitable subtraction and using Lemma 4 (d), and $\frac{1}{T}\left\|\tilde{f}_t-H_1^\top f_t\right\|^4 = O_p\left(\frac{1}{\delta_{NT}^4}\right)$ by Lemma 4 (a).

Next, $\frac{1}{T_1}\sum_{t=1}^{T} E.II_t^4$ can be bounded by:

$$\begin{aligned}\frac{1}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor} E.II_t^4 &\leq \left\|\tilde{\lambda}_{1,i}^{\top} - \lambda_{1,i}^{\top} H_1^{-\top}\right\|^4 \|H_1\|^4 \frac{1}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor} \|f_t\|^4 \\ &= O_p\left(\frac{1}{\delta_{NT}^4}\right) O_p(1)\, O_p(1),\end{aligned}$$

where $\left\|\tilde{\lambda}_{1,i}^{\top} - \lambda_{1,i}^{\top} H_1^{-\top}\right\|^4 = O_p\left(\frac{1}{\delta_{NT}^4}\right)$ by Lemma 4 (d), and $\frac{1}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor} \|f_t\|^4 = O_p(1)$ by Assumption 1.

The second part of this lemma follows immediately, because

$$\begin{aligned}\sup_{\pi \in [\pi_1, \pi_2]} \left\{ \frac{1}{\pi T}\sum_{t=1}^{\lfloor \pi T \rfloor} |\widehat{e}_{it} - e_{it}| \right\} &\leq \frac{1}{\pi_1 T}\sum_{t=1}^{T} |\widehat{e}_{it} - e_{it}|^4 \\ &= \frac{1}{\pi_1 T}\sum_{t=1}^{\lfloor \pi T \rfloor} \left|\tilde{e}_{(1),it} - e_{it}\right|^4 + \frac{1}{\pi_1 T}\sum_{t=\lfloor \pi T + 1 \rfloor}^{T} \left|\tilde{e}_{(2),it} - e_{it}\right|^4 \\ &= o_p(1).\end{aligned}$$

The third part of this lemma also holds following a similar argument. ■

Proof of Lemma 10 (b).

The proof is similar to that of Lemma 10 (a) - it suffices to show that $\frac{1}{T_1 N}\sum_{t=1}^{T}\sum_{i=1}^{N} E.I_t^4$ and $\frac{1}{T_1 N}\sum_{t=1}^{T}\sum_{i=1}^{N} E.II_t^4$ are both negligible. For brevity, we will focus on the case of

$m = 1$, as the case for $m = 2$ is similar. First, $\frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N} E.II_t^4$ can be bounded by

$$
\begin{aligned}
&\frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N} E.I_t^4 \\
=&\frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N} \tilde{\lambda}_{1,i}^{\top}(\tilde{f}_t - H_1^{\top} f_t) \\
=&\frac{1}{T_1 N}\sum_{i=1}^{N}\sum_{t=1}^{\lfloor \pi T \rfloor} \left[\left(\tilde{\lambda}_{1,i} - \lambda_{1,i} H_1^{-\top}\right)\left(\tilde{f}_t - H_1^{\top} f_t\right) + \lambda_{1,i} H_1^{\top}\left(\tilde{f}_t - H_1^{\top} f_t\right)\right]^4 \\
\leq&\frac{64}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\left(\tilde{f}_t - H_1^{\top} f_t\right)^4 \frac{1}{N}\left(\tilde{\lambda}_{1,i} - \lambda_i H_1^{-\top}\right)^4 + \frac{64}{T_1 N}\sum_{i=1}^{N}\left(\lambda_{1,i} H_1^{\top}\right)^4 \sum_{t=1}^{\lfloor \pi T \rfloor}\left(\tilde{f}_t - H_1^{\top} f_t\right)^4 \\
\leq&\frac{64}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor}\left\|\tilde{f}_t - H_1^{\top} f_t\right\|^4 \frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{1,i} - \lambda_{1,i} H_1^{-\top}\right\|^4 + 64\frac{1}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor}\left\|\tilde{f}_t - H_1^{\top} f_t\right\|^4 \frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1,i}\|^4 \|H_1\|^4 \\
=&o_p(1)o_p(1) + o_p(1)O_p(1) \\
=&o_p(1),
\end{aligned}
$$

where the first $o_p(1)$ term comes from Lemmas 4 (a) and 4 (d), and the second term is $o_p(1)$ from Lemmas 4 (a) and 1 (e) and Assumption 2. The second term $\frac{1}{T_1 N}\sum_{t=1}^{T}\sum_{i=1}^{N} E.II_t^4$ can be bounded by

$$
\begin{aligned}
\frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N} E.II_t^4 &= \frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N}\left(\tilde{\lambda}_{1,i}^{\top} - \lambda_{1,i}^{\top} H_1^{-\top}\right) H_1^{\top} f_t \\
&\leq \frac{1}{T_1 N}\sum_{t=1}^{\lfloor \pi T \rfloor}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{1,i} - \lambda_{1,i} H_1^{-\top}\right\|^4 \|H_1\|^4 \|f_t\|^4 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{1,i} - \lambda_{1,i} H_1^{-\top}\right\|^4 \frac{1}{T_1}\sum_{t=1}^{\lfloor \pi T \rfloor}\|f_t\|^4 \|H_1\|^4 \\
&= o_p(1)O_p(1)\,O_p(1) \\
&= o_p(1),
\end{aligned}
$$

because of Lemmas 4 (d) and 1 (e) and Assumption 1.

The second part of this lemma follows immediately because

$$\begin{aligned}\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi T}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{it}-e_{it}\right|^4\right\} &\leq \frac{1}{\pi_1 T}\sum_{t=1}^{T}\left|\widehat{e}_{it}-e_{it}\right|^4 \\ &= \frac{1}{\pi_1 T}\sum_{t=1}^{\lfloor\pi T\rfloor}\left|\widehat{e}_{(1),it}-e_{it}\right|^4+\frac{1}{\pi_1 T}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\left|\widehat{e}_{(2),it}-e_{it}\right|^4 \\ &= O_p\left(\frac{1}{\delta_{NT}^2}\right).\end{aligned}$$

The third part also similarly follows. ■

Proof of Lemmas 10 (c) and 10 (d).

Lemmas 10 (c) and 10 (d) are implications of Lemmas 10 (a) and 10 (b) and can be proven in a similar way.

For $m=1$ (noting that the case for $m=2$ is similar), the lemma holds as

$$\frac{1}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}\tilde{e}_{(1),it}^4 \leq \frac{8}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\tilde{e}_{(1),it}-e_{it}\right)^4+\frac{8}{T_1}\sum_{t=1}^{\lfloor\pi T\rfloor}e_{it}^4 = O_p\left(1\right).$$

The second part of this lemma follows as

$$\begin{aligned}\sup_{\pi\in[\pi_1,\pi_2]}\left\{\frac{1}{\pi T}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{e}_{it}^4\right\} &\leq \frac{1}{\pi_1 T}\sum_{t=1}^{T}\widehat{e}_{it}^4 \\ &= \frac{1}{\pi_1 T}\sum_{t=1}^{\lfloor\pi T\rfloor}\widehat{e}_{(1),it}^4+\frac{1}{\pi_1 T}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\widehat{e}_{(2),it}^4 = O_p\left(1\right).\end{aligned}$$

The third part follows similarly.

Lemma 10 (d) can be proved similarly to Lemma 10 (c). ■

Next, we define the covariance matrices $\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)$ and $\tilde{\Theta}_{2,i}\left(\pi,\widehat{F}\right)$. These were discussed in Bai (2003) and are HAC estimators constructed using the estimated residuals $\tilde{e}_{(1),it}=x_{it}-\tilde{\lambda}_{2,i}^{\top}\tilde{f}_t$ and $\tilde{e}_{(2),it}=x_{it}-\tilde{\lambda}_{2,i}^{\top}\tilde{f}_{2,t}$ in the series $\tilde{Z}^{\top}\tilde{f}_t\cdot\tilde{e}_{(1),it}$ and $\tilde{f}_{2,t}\cdot\tilde{e}_{(2),it}$

respectively:

$$\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)=\widehat{D}_{0,1,i}\left(\pi,\widehat{F}\right)+\sum_{j=1}^{\lfloor\pi T\rfloor-1}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\left(\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)+\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)^{\top}\right), \quad \text{(A.10)}$$

$$\tilde{\Theta}_{2,i}\left(\pi,\widehat{F}\right)=\widehat{D}_{0,2,i}\left(\pi,\widehat{F}\right)+\sum_{j=1}^{T-\lfloor\pi T\rfloor-1}\mathbf{k}\left(\frac{j}{b_{T-\lfloor\pi T\rfloor}}\right)\left(\widehat{D}_{j,2,i}\left(\pi,\widehat{F}\right)+\widehat{D}_{j,2,i}\left(\pi,\widehat{F}\right)^{\top}\right), \quad \text{(A.11)}$$

where

$$\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)=\frac{1}{\pi T}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\widehat{f}_t\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\widehat{f}_{t-j}^{\top}, \quad \text{(A.12)}$$

$$D_{j,2,i}\left(\pi,\widehat{F}\right)=\frac{1}{(1-\pi)T}\sum_{t=\lfloor\pi T\rfloor+j+1}^{T}\widehat{f}_t\tilde{e}_{(2),it}\tilde{e}_{(2),i,t-j}\widehat{f}_{t-j}^{\top}, \quad \text{(A.13)}$$

and $\mathbf{k}(.)$ is a real valued kernel, such as the Bartlett kernel, satisfying Assumption 9. Additionally, define the HAC estimators using infeasible data as $\tilde{\Theta}_{1,i}(\pi, F_{0,1})$ and $\tilde{\Theta}_{2,i}(\pi, FH_{0,2})$, and assume that they are consistent for $\Theta_{1,i}$ and $\Theta_{2,i}$, respectively.

Lemma 11. *Under Assumptions 1 to 9, for $m = 1, 2$, we have*

(a) $\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\widehat{D}_{j,m,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,m,i}\left(\pi,FH_{0,m}\right)\right\|=o_p(1),$

(b) $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Theta}_{m,i}\left(\pi,\widehat{F}\right)-\Theta_{m,i}\left(\pi,FH_{0,m}\right)\right\|=o_p(1).$

Proof of Lemma 11 (a).

We focus on the case of $m = 1$, as the proof for $m = 2$ is similar and thus omitted. Due to the consistency of H_m for $H_{0,m}$, it suffices to show

$$\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\|\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1,i}\left(\pi,FH_1\right)\right\|=o_p(1).$$

We have

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1}\left(\pi,FH_1\right)\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\tilde{f}_t\tilde{f}_{t-j}^{\top}\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-H_1^{\top}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}f_tf_{t-j}^{\top}e_{it}e_{i,t-j}\right)H_1\right\| \\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right) \\
&+\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(H_1^{\top}f_tf_{t-j}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right) \\
=&D.I+D.II.
\end{aligned}
$$

The first term $D.I$ is bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left\{\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}^{\top}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)\right\}\\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}^{\top}H_1\right\|^2\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)^2\right)^{\frac{1}{2}}\\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\left(\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right)+\left(\tilde{f}_t-H_1^{\top}f_t\right)\tilde{f}_{t-j}^{\top}\right\|^2\right.\\
&\qquad\left.\times\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)^2\right)^{\frac{1}{2}}\\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left(\frac{2}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\left(\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right)\right\|^2+\left\|\left(\tilde{f}_t-H_1^{\top}f_t\right)\left(\tilde{f}_{t-j}^{\top}\right)\right\|^2\right)^{\frac{1}{2}}\\
&\qquad\times\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)\right)^{\frac{1}{2}}\\
\le&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\le j\le\lfloor\pi T\rfloor-1}\left(2\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\right\|^4\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right\|^4\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+2\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_{t-j}\right\|^4+\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t-H_1^{\top}f_t\right\|^4\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\\
&\qquad\times\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\tilde{e}_{(1),it}^2\tilde{e}_{(1),i,t-j}^2\right)^{1/2}\\
\le&o_p(1)\left(\frac{1}{\lfloor\pi_1T\rfloor}\sum_{t=1}^{T}\tilde{e}_{(1),it}^2\tilde{e}_{(1),it}^2\right)^{1/2}\\
\le&o_p(1)\left[\left(\frac{1}{\lfloor\pi_1T\rfloor}\sum_{t=1}^{T}\tilde{e}_{(1),it}^4\right)^{\frac{1}{2}}\left(\frac{1}{\lfloor\pi_1T\rfloor}\sum_{t=1}^{T}\tilde{e}_{(1),it}^4\right)^{\frac{1}{2}}\right]^{1/2}\\
=&o_p(1)O_p\left(1\right),
\end{aligned}
$$

where the first term being $o_p(1)$ follows from applying Lemma 4 (a), and the second term

follows from Lemma 10 (d). Using Lemma 4 (c) to replace H_1 with $H_{0,m}$ results in a convergence speed of $O_p\left(\delta_{NT}^{-1}\right)$.

The second term $D.II$ is bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(H_1^\top f_t f_{t-j}^\top H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right)\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_t f_{t-j}^\top H_1\right\|^2\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right)\right)^2\right)^{\frac{1}{2}}\\
\leq&O_p\left(1\right)\left(\frac{2}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{T}\left(\tilde{e}_{(1),it}^2\left(\tilde{e}_{(1),it}-e_{it}\right)^2+e_{it}^2\left(\tilde{e}_{(1),it}-e_{it}\right)^2\right)\right)^{\frac{1}{2}}\\
\leq&O_p\left(1\right)\left(\frac{2}{\lfloor\pi_1 T\rfloor}\left(\sum_{t=1}^{T}\tilde{e}_{(1),it}^4\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\tilde{e}_{(1),i,t}-e_{it}\right)^4\right)^{\frac{1}{2}}+\frac{2}{\lfloor\pi_1 T\rfloor}\left(\sum_{t=1}^{T}e_{it}^4\sum_{t=1}^{T}\left(\tilde{e}_{(1),it}-e_{it}\right)^4\right)^{\frac{1}{2}}\right)\\
=&o_p(1),
\end{aligned}
$$

where the $O_p\left(1\right)$ term comes from Assumption 1 and Lemma 1 (e), and the $o_p(1)$ term comes from Lemma 10 (b). ■

Proof of Lemma 11 (b).

We focus on the case of $m=1$, as the proof for $m=2$ is similar and thus omitted. See that

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)-\Theta_{1,i}\left(\pi,FH_{0,1}\right)\right\|\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Theta}_{m,i}\left(\pi,\widehat{F}\right)-\Theta_{1,i}\left(\pi,FH_1\right)\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Theta}_{1,i}\left(\pi,FH_1\right)-\Theta_{1,i}\left(\pi,FH_{0,1}\right)\right\|\\
=&\Theta_{w_i}.I+\Theta_{w_i}.II.
\end{aligned}
$$

For the term $\Theta_{w_i}.I$ we have:

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)-\Theta_{1,i}\left(\pi,FH_1\right)\right\| \\
\leq &\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{D}_{0,1,i}\left(\pi,\widehat{F}\right)+\sum_{j=1}^{\lfloor\pi T\rfloor-1}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\left(\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)+\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)^{\top}\right)\right. \\
&\left.-\widehat{D}_{0,1,i}(\pi,FH_1)-\sum_{j=1}^{\lfloor\pi T\rfloor-1}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\left(\widehat{D}_{j,1,i}(\pi,FH_1)+\widehat{D}_{j,1,i}(\pi,FH_1)^{\top}\right)\right\|.
\end{aligned}$$

Recall that $\left|\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\right|\leq 1$ and $\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)=0$ if $j>b_{\lfloor\pi T\rfloor}$ for the Bartlett kernel. Thus,

$$\Theta_{w_i}.I\leq\sup_{\pi\in[\pi_1,\pi_2]}\left\|\widehat{D}_{0,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{0,1,i}(\pi,FH_1)\right\|+2\sup_{\pi\in[\pi_1,\pi_2]}\sum_{j=1}^{b_{\lfloor\pi T\rfloor}}\left\|\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1,i}(\pi,FH_1)\right\|.$$

In the case of the Bartlett kernel, $\Theta_{w_i}.I$ is $O_p\left(\frac{T^{1/3}}{\delta_{NT}}\right)$ by Lemma 11 (a) and the condition in Assumption 9 (b), which states that $b_{\lfloor\pi T\rfloor}=O\left(T^{1/3}\right)$, so $\Theta_{w_i}.I$ is $o_p(1)$ if $b_T/\delta_{NT}\to 0$ as $N,T\to\infty$.

The term $\Theta_{w_i}.II$ can be also be shown to be $o_p(1)$ with similar arguments. ■

Lemma 12. *Under Assumptions 1 to 9 and Assumptions W1 to W3, for $m=1,2$,*

(a) $\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,m,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,m}\left(\pi,FH_{0,m}\right)\right\|=o_p(1)$, *where* $\widehat{D}_{j,m}\left(\pi,FH_{0,m}\right)=\operatorname{plim}\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,m,i}\left(\pi,FH_{0,m}\right)$, *its infeasible counterpart,*

(b) $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{\Theta}_{m,i}\left(\pi,\widehat{F}\right)-\tilde{\Theta}_{m,i}\left(\pi,FH_{0,m}\right)\right\|=o_p(1)$.

Proof of Lemma 12 (a).

We focus on the case of $m=1$, as the proof for $m=2$ is similar and thus omitted. Similar to Lemma 11 (a), due to the consistency of H_1 to $H_{0,1}$, it suffices to show

$$\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1}\left(\pi,FH_1\right)\right\|=o_p(1).$$

We have

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1}\left(\pi,FH_1\right)\right\| \\
=&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\|\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\tilde{f}_t\tilde{f}_{t-j}^{\top}\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right. \\
&\qquad\left.\times H_1^{\top}\left(\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}f_tf_{t-j}^{\top}e_{it}e_{i,t-j}\right)H_1\right\| \\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right) \\
&+\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(H_1^{\top}f_tf_{t-j}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right) \\
=&D.III+D.IV.
\end{aligned}
$$

The first term $D.III$ is bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left\{\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}^{\top}H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)\right\}\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\tilde{f}_{t-j}^{\top}-H_1^{\top}f_tf_{t-j}^{\top}H_1\right\|^2\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)^2\right)^{\frac{1}{2}}\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\left(\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right)+\left(\tilde{f}_t-H_1^{\top}f_t\right)\left(\tilde{f}_{t-j}^{\top}\right)\right\|^2\right.\\
&\qquad\left.\times\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)^2\right)^{\frac{1}{2}}\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{2}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\left(\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right)\right\|^2+\left\|\left(\tilde{f}_t-H_1^{\top}f_t\right)(\tilde{f}_{t-j}^{\top})\right\|^2\right)^{\frac{1}{2}}\\
&\qquad\times\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}\right)\right)^{\frac{1}{2}}\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(2\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t\right\|^4\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_{t-j}^{\top}-f_{t-j}^{\top}H_1\right\|^4\right)^{\frac{1}{2}}\right.\\
&\qquad\left.+2\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_{t-j}\right\|^4+\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|\tilde{f}_t-H_1^{\top}f_t\right\|^4\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\\
&\qquad\times\left(\frac{1}{\lfloor\pi T\rfloor N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\sum_{i=1}^{N}\tilde{e}_{(1),it}^2\tilde{e}_{(1),i,t-j}^2\right)^{1/2}\\
\leq&o_p(1)\left(\frac{1}{\lfloor\pi_1T\rfloor N}\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{e}_{(1),it}^2\tilde{e}_{(1),it}^2\right)^{1/2}\\
\leq&o_p(1)\left(\frac{1}{\lfloor\pi_1T\rfloor N}\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{e}_{(1),it}^4\right)^{1/2}\\
=&o_p(1)O_p\left(1\right),
\end{aligned}
$$

where the first term is $o_p(1)$ follows from applying Lemma 4 (a), and the second term follows from Lemma 10 (d). Replacing H_1 with $H_{0,1}$ using Lemma 4 (c) results in a convergence

rate of $O_p\left(\delta_{NT}^{-1}\right)$.

The second term $D.IV$ is bounded by

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\frac{1}{\lfloor\pi T\rfloor N}\sum_{i=1}^{N}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(H_1^\top f_t f_{t-j}^\top H_1\right)\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right)\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\sup_{0\leq j\leq\lfloor\pi T\rfloor-1}\left(\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left\|H_1^\top f_t f_{t-j}^\top H_1\right\|^2\frac{1}{\lfloor\pi T\rfloor}\sum_{t=j+1}^{\lfloor\pi T\rfloor}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{e}_{(1),it}\tilde{e}_{(1),i,t-j}-e_{it}e_{i,t-j}\right)\right)^2\right)^{\frac{1}{2}}\\
\leq&O_p\left(1\right)\left(\frac{2}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{e}_{(1),it}^2\left(\tilde{e}_{(1),it}-e_{it}\right)^2+e_{it}^2\left(\tilde{e}_{(1),it}-e_{it}\right)^2\right)\right)^{\frac{1}{2}}\\
\leq&O_p\left(1\right)\left(\frac{2}{\lfloor\pi_1 T\rfloor N}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{e}_{(1),it}^4\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\tilde{e}_{(1),it}-e_{it}\right)^4\right)^{\frac{1}{2}}\right.\\
&\left.+\frac{2}{\lfloor\pi_1 T\rfloor N}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}e_{it}^4\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\tilde{e}_{(1),it}-e_{it}\right)^4\right)^{\frac{1}{2}}\right)\\
=&o_p(1),
\end{aligned}
$$

where the $O_p\left(1\right)$ term comes from Assumption 1 and Lemma 1 (e), and the $o_p(1)$ term comes from Lemma 10 (b). ■

Proof of Lemma 12 (b).

For brevity, we will only prove the case for $m=1$, as the case for $m=2$ is similar. See that

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)-\tilde{\Theta}_{1,i}\left(\pi,FH_{0,1}\right)\right\|\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)-\tilde{\Theta}_{1,i}\left(\pi,FH_{1}\right)\right\|+\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{\Theta}_{1,i}\left(\pi,FH_{1}\right)-\tilde{\Theta}_{1,i}\left(\pi,FH_{0,1}\right)\right\|\\
=&\Theta_W.I+\Theta_W.II.
\end{aligned}
$$

For the term $\Theta_W.I$, we have:

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{\Theta}_{1,i}\left(\pi,\widehat{F}\right)-\tilde{\Theta}_{1,i}\left(\pi,FH_1\right)\right\| \\
\leq &\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}D_{0,1,i}\left(\pi,\widehat{F}\right)+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\lfloor\pi T-1\rfloor}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)\right. \\
&\left.-\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,1,i}(\pi,FH_1)-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\lfloor\pi T-1\rfloor}\mathbf{k}\left(\frac{j}{b_{\lfloor\pi T\rfloor}}\right)\widehat{D}_{j,1,i}(\pi,FH_1)\right\| \\
\leq &\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{0,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{0,1,i}(FH_1)\right\|+2\sup_{\pi\in[\pi_1,\pi_2]}\sum_{j=1}^{b_{\lfloor\pi T\rfloor}}\left\|\frac{1}{N}\sum_{i=1}^{N}\widehat{D}_{j,1,i}\left(\pi,\widehat{F}\right)-\widehat{D}_{j,1,i}(FH_1)\right\| \\
=&o_p(1),
\end{aligned}$$

by Lemma 12 (a), and the condition that $b_T/\delta_{NT}\to 0$ in Assumption 9 (b). The term $\Theta_W.II$ can be shown to be $o_p(1)$ using similar arguments. ■

Lemma 13. *Under Assumptions 1 to 8, if* $\sqrt{T}/N\to 0$*,*

(a)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_2^{-1}w_i\right)^{-1}\right\|=O_p\left(1\right),\ \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}\right\|=O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}-\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|=o_p(1).$$

(b)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|=O_p\left(1\right),\ \sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W}\left(\pi,\tilde{w}_i\right)^{-1}\right\|=O_p\left(1\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W}\left(\pi,\tilde{w}_i\right)^{-1}-\tilde{\Omega}_{W}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|=o_p(1).$$

Proof of Lemma 13.

For the first part of Lemma 13 (a), Assumption 9 and the fact that $0<\pi_1\leq\pi\leq\pi_2<1$

imply that $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)-\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_{W,i}\right\|=o_p(1)$. Because $\Omega_{W,i}$ is positive definite,

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left|\rho_{min}\left(\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)\right)-\rho_{min}\left(\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_{W,i}\right)\right|\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)-\left(\frac{1}{\pi}+\frac{1}{1-\pi}\right)\Omega_{W,i}\right\|\\
=&o_p(1),
\end{aligned}$$

which means that the eigenvalues of $\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)$ are bounded away from 0 uniformly in π, and thus $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|=O_p\left(1\right)$. Following a similar argument, the eigenvalues of $\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)$ are also bounded away from zero uniformly in π, implying that $\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}\right\|=O_p\left(1\right)$ as required.

For the second part of Lemma 13 (a):

$$\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}-\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|\\
=&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\left(\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)-\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)\right)\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}\right\|\\
\leq&\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)^{-1}\right\|\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,H_{0,2}^{-1}w_i\right)-\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)\right\|\sup_{\pi\in[\pi_1,\pi_2]}\left\|\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1}\right\|\\
=&o_p(1).
\end{aligned}$$

Lemma 13 (b) can be proven in a similar way to Lemma 13 (a). ■

Proof of Theorem 3.3.

For Theorem 3.3 (a), note that

$$\begin{aligned}\mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1}w_i\right) \equiv & \left[\frac{1}{(1-\pi)\sqrt{T}}H_{0,2}^{\top}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} - \frac{1}{\pi\sqrt{T}}H_{0,2}^{\top}Z^{\top}H_{0,1}^{\top}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it}\right]^{\top} \\ & \times \Omega_{W,i}\big(\pi, H_{0,2}^{-1}w_i\big)^{-1} \\ & \times \left[\frac{1}{(1-\pi)\sqrt{T}}H_{0,2}^{\top}\sum_{t=\lfloor \pi T+1\rfloor}^{T} f_t e_{it} - \frac{1}{\pi\sqrt{T}}H_{0,2}^{\top}Z^{\top}H_{0,1}H_{0,1}^{\top}\sum_{t=1}^{\lfloor \pi T\rfloor} f_t e_{it}\right] \\ = & \operatorname{plim}\left(W.I + W.II\right), \end{aligned} \tag{A.14}$$

i.e. the main dominating terms in Lemma 9 (a). Recalling that $H_2 \xrightarrow{p} H_{0,2}$, it thus suffices to show

$$\sup_{\pi\in[\pi_1,\pi_2]}\left|\mathscr{W}_{W,i}\left(\pi,\tilde{w}_i\right) - \mathscr{W}_{W,i}\left(\pi, H_2^{-1}w_i\right)\right| = o_p(1).$$

We have

$$\begin{aligned} & \sup_{\pi\in[\pi_1,\pi_2]}\left|\mathscr{W}_{W,i}\left(\pi,\tilde{w}_i\right) - \mathscr{W}_{W,i}\left(\pi, H_2^{-1}w_i\right)\right| \\ \leq & \sup_{\pi\in[\pi_1,\pi_2]}\left|\sqrt{T}(\tilde{w}_i)^{\top}\left[\tilde{\Omega}_{W,i}\left(\pi,\tilde{w}_i\right)^{-1} - \tilde{\Omega}_{W,i}\left(\pi, H_{0,2}^{-1}w_i\right)^{-1}\right]\sqrt{T}(\tilde{w}_i)\right| \\ & + \sup_{\pi\in[\pi_1,\pi_2]}\left|\sqrt{T}(\tilde{w}_i - W.I - W.II)^{\top}\tilde{\Omega}_{W,i}\left(\pi, H_{0,2}^{-1}w_i\right)^{-1}\sqrt{T}(\tilde{w}_i)^{\top}\right| \\ & + \sup_{\pi\in[\pi_1,\pi_2]}\left|\sqrt{T}(\tilde{w}_i)^{\top}\tilde{\Omega}_{W,i}\left(\pi, H_{0,2}^{-1}w_i\right)^{-1}\sqrt{T}(\tilde{w}_i - W.I - W.II)^{\top}\right| \\ = & o_p(1), \end{aligned}$$

where we have used the fact that $\sqrt{T}(\tilde{w}_i - W.I - W.II)^{\top} = O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)$ following Lemma 9 (a) under the case of the null hypothesis where $w_i = 0$, and the results of Lemma 13 (a).

Because $\mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \Rightarrow Q_r(\pi)$ by Assumption W4 (b), it is sufficient to show that

$$\left| \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) - \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \right| = o_p(1).$$

By adding and subtracting and taking the supremum on both sides, we have

$$\sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) \leq \left| \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) - \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \right| + \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right).$$

Therefore,

$$\sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) - \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \leq \sup_{\pi \in [\pi_1, \pi_2]} \left| \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) - \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \right| = o_p(1).$$

Similarly, it can be shown that

$$\sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) - \sup_{\pi \in [\pi_1, \pi_2]} \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) \leq \sup_{\pi \in [\pi_1, \pi_2]} \left| \mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) - \mathscr{W}_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right) \right| = o_p(1).$$

Combining these inequalities yields the desired result.

Theorem 3.3 (b) can be proven in a similar way using Lemma 13 (b). ■

Proof of Theorem 3.4.

For Theorem 3.4 (a), we will show that $\mathscr{W}_{W,i}\left(\pi, \tilde{w}_i\right) \to \infty$ as $N, T \to \infty$ when $w_i \neq 0$. Recalling the asymptotic expansion of $\tilde{w}_i$, we have:

$$\begin{aligned}
\tilde{w}_i =& (H_2^{-1} w_i) + H_2^{-1} \frac{1}{(1-\pi)T} \sum_{t=\lfloor \pi T + 1 \rfloor}^{T} f_t e_{it} - \tilde{Z}^\top \frac{1}{\pi T} \sum_{t=1}^{\lfloor \pi T \rfloor} H_1^\top f_t e_{it} + O_p\left(\frac{1}{\delta_{NT}^2}\right) \\
=& H_2^{-1} w_i + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\delta_{NT}^2}\right), \\
=& H_2^{-1} w_i + o_p(1).
\end{aligned}$$

Because $\tilde{\Omega}_{W,i}(\pi, \tilde{w}_i) \xrightarrow{p} \Omega_{W,i}\left(\pi, H_2^{-1} w_i\right)$ and $\Omega_{W,i}\left(\pi, H_2^{-1} w_i\right)$ is positive definite, it follows

that $\tilde{\Omega}_{W,i}(\pi,\tilde{w}_i)^{-1} = O_p(1)$ and we have the desired result:

$$\text{plim}_{N,T\to\infty} \inf\left(\mathscr{W}_{W_i}(\pi,\tilde{w}_i)\right) = \text{plim}_{N,T\to\infty} \inf\left(\tilde{w}_i^\top \Omega_{W,i}\left(\pi, H_{0,2}^{-1} w_i\right)^{-1} \tilde{w}_i\right) > 0,$$

which implies the desired divergence under the alternative hypothesis. The consistency of $\sup_{\pi\in[\pi_1,\pi_2]} \mathscr{W}_{W,i}(\pi,\tilde{w}_i)$ follows from the consistency of $\mathscr{W}_{W,i}(\pi,\tilde{w}_i)$.

For Theorem 3.4 (b), we will show that $\mathscr{W}_W(\pi,\tilde{w}_i) \to \infty$ as $N,T \to \infty$ under the alternative. We have:

$$\begin{aligned}
&(T^{\alpha/2})\left(\frac{\sum_{i=1}^N \tilde{w}_i}{\sqrt{N}}\right) \\
=&T^{\alpha/2}\frac{\sum_{i=1}^N (H_2^{-1}w_i)}{\sqrt{N}} + H_2^\top \frac{T^{\alpha/2}}{(1-\pi)T\sqrt{N}} \sum_{i=1}^N \sum_{s=\lfloor \pi T+1\rfloor}^{T} f_s e_{is} \\
&- \tilde{Z}^\top \frac{T^{\alpha/2}}{\pi T\sqrt{N}} \sum_{i=1}^N \sum_{s=1}^{\lfloor \pi T\rfloor} H_1^\top f_s e_{is} + O_p\left(\frac{T^{\alpha/2}}{\delta_{NT}^2}\right) \\
=&H_2^\top T^{\alpha/2}\frac{\sum_{i=1}^N (H_2^{-1}w_i)}{\sqrt{N}} + H_2^\top \frac{1}{(1-\pi)T^{1-\alpha/2}} O_p(1) - \tilde{Z}^\top H_1^\top \frac{1}{\pi T^{1-\alpha/2}} O_p(1) + O_p\left(\frac{T^{\alpha/2}}{\delta_{NT}^2}\right) \\
=&H_2^\top T^{\alpha/2}\frac{\sum_{i=1}^N (H_2^{-1}w_i)}{\sqrt{N}} + O_p\left(\frac{1}{T^{1-\alpha/2}}\right) + O_p\left(\frac{T^{\alpha/2}}{\delta_{NT}^2}\right),
\end{aligned}$$

where the last two terms are $o_p(1)$ because $0 < \alpha \le 0.5$ and $\sqrt{T}/N \to 0$ as $N,T \to \infty$. Since $\tilde{\Omega}_W(\pi,\tilde{w}_i) \xrightarrow{p} \Omega_W$ and Ω_W is positive definite, it follows that $\tilde{\Omega}_W(\pi,\tilde{w}_i)^{-1} = O_p(1)$, and we have the desired result by Assumption W5:

$$\begin{aligned}
&\text{plim}_{N,T\to\infty} \inf\left(\frac{T^\alpha}{T}\mathscr{W}_W(\pi,\tilde{w}_i)\right) \\
=&\,\text{plim}_{N,T\to\infty} \inf\left[(T^\alpha)\left(\frac{\sum_{i=1}^N \tilde{w}_i}{\sqrt{N}}\right)^\top \tilde{\Omega}_W(\pi,\tilde{w}_i)^{-1}\left(\frac{\sum_{i=1}^N \tilde{w}_i}{\sqrt{N}}\right)\right] > 0,
\end{aligned}$$

which implies the desired divergence under the alternative hypothesis. The consistency of $\sup_{\pi\in[\pi_1,\pi_2]} \mathscr{W}_W(\pi,\tilde{w}_i)$ follows from the consistency of $\mathscr{W}_W(\pi,\tilde{w}_i)$. ■

A.5 Estimated Break Fraction

In this section, we study the effects of using an estimated break fraction as discussed in Section 3.4. Define the true breakpoint as $k = \lfloor \pi T \rfloor$, and its estimator as $\widehat{k} = \lfloor \widehat{\pi} T \rfloor$, such that $|\widehat{\pi} - \pi| = O_p(T^{-1})$, i.e. $\left|\widehat{k} - k\right| = O_p(1)$, which can be easily satisfied by many existing estimators, such as that of Baltagi et al. (2017). We proceed by splitting the analysis into two cases: 1) $\widehat{k} \leq k$, and 2) $\widehat{k} > k$. Due to symmetry, it suffices to study the behavior of $\tilde{F}_1(\widehat{k})$, i.e. the subsample estimate of pre-break factors. In the first case, the break is under-/exactly specified, and hence the pre-break partition of data based on this can be viewed as a sample with no structural break. Thus, the pre-existing results for the case of a stable factor model apply.

In the second case, the break is over-specified, which implies that a partition of data based on this experiences a structural break, and hence pre-existing results in the literature are not directly applicable. To overcome this, we will treat the terms related to the structural change as an extra error term, such that

$$\begin{aligned} x_{it} &= f_t^\top \lambda_{2i} + e_{it} \\ &= f_t^\top \lambda_{1i} + e_{it} + u_{it} \\ &= a_{it} + u_{it}, \end{aligned} \tag{A.15}$$

where

$$a_{it} = f_t^\top \lambda_{1i} + e_{it}, \tag{A.16}$$

$$u_{it} = \begin{cases} 0, & t = 1, \ldots, k, \\ f_t^\top \lambda_{2i} - f_t^\top \lambda_{1i}, & t = k+1, \ldots, T. \end{cases} \tag{A.17}$$

That is, when $\widehat{k} > k + 1$, the pre-break subsample can be treated as a data partition with

no structural change, but with an extra error term. In matrix form, we have

$$\begin{aligned} X_1(\widehat{k}) &= A(\widehat{k}) + U(\widehat{k}), \\ &= F_1(\widehat{k})\Lambda_1^\top + e_{(1)}(\widehat{k}) + U(\widehat{k}), \end{aligned} \tag{A.18}$$

$$X_2(\widehat{k}) = F_2(\widehat{k})\Lambda_2^\top + e_{(2)}(\widehat{k}), \tag{A.19}$$

where $X_1(\widehat{k})$, $A(\widehat{k})$, $e_{(1)}(\widehat{k})$, and $U(\widehat{k})$ are all $\widehat{k} \times N$ matrices, and $X_2(\widehat{k})$, $e_{(2)}(\widehat{k})$, are all $\left(T - \widehat{k}\right) \times N$ matrices. Thus,

$$\begin{aligned} X_1(\widehat{k})X_1(\widehat{k})^\top =& F_1(\widehat{k})\Lambda_1^\top\Lambda_1 F_1^\top(\widehat{k}) + F_1(\widehat{k})\Lambda_1^\top\left[e_{(1)}(\widehat{k}) + U(\widehat{k})\right]^\top \\ &+ \left[e_{(1)}(\widehat{k}) + W(\widehat{k})\right]\Lambda_1 F_1(\widehat{k}) + \left[e_{(1)}(\widehat{k}) + W(\widehat{k})\right]\left[e_{(1)}(\widehat{k}) + U(\widehat{k})\right]^\top. \end{aligned} \tag{A.20}$$

Define $H_1(\widehat{k}) = \left(\frac{\Lambda_1^\top\Lambda_1}{N}\right)\left(\frac{F_1(\widehat{k})^\top\tilde{F}_1(\widehat{k})}{\widehat{k}}\right)V_{NT,1}(\widehat{k})^{-1}$, the counterpart of H_1. It follows that

$$\begin{aligned} \tilde{F}_1(\widehat{k}) - FH_1(\widehat{k}) = \frac{1}{N\widehat{k}}\Big[& e_{(1)}(\widehat{k})\Lambda_1 F_1(\widehat{k})^\top\tilde{F}_1(\widehat{k}) + F_1(\widehat{k})\Lambda_1^\top e_{(1)}(\widehat{k})^\top\tilde{F}_1(\widehat{k}) + e_{(1)}(\widehat{k})e_{(1)}(\widehat{k})^\top\tilde{F}_1(\widehat{k}) \\ &+ U(\widehat{k})^\top\Lambda_1 F_1(\widehat{k})\tilde{F}_1(\widehat{k}) + F_1(\widehat{k})\Lambda_1^\top U(\widehat{k})^\top\tilde{F}_1(\widehat{k}) + U(\widehat{k})U(\widehat{k})^\top\tilde{F}_1(\widehat{k}) \\ &+ U(\widehat{k})e_{(1)}(\widehat{k})^\top\tilde{F}_1(\widehat{k}) + e_{(1)}(\widehat{k})U(\widehat{k})^\top\tilde{F}_1(\widehat{k})\Big]V_{NT,1}(\widehat{k})^{-1}, \end{aligned} \tag{A.21}$$

$$\begin{aligned} \tilde{f}_t(\widehat{k}) - H_1(\widehat{k})^\top f_t =& V_{NT,1}(\widehat{k})^{-1}\frac{1}{N\widehat{k}}\Big[\tilde{F}_1(\widehat{k})^\top F_1(\widehat{k})\Lambda_1^\top e_t + \tilde{F}_1(\widehat{k})^\top E_1(\widehat{k})\Lambda_1 f_t + \tilde{F}_1(\widehat{k})^\top E_1(\widehat{k})e_t \\ &+ \tilde{F}_1(\widehat{k})^\top F_1(\widehat{k})\Lambda_1 u_t + \tilde{F}_1(\widehat{k})^\top U(\widehat{k})\Lambda_1 f_t + \tilde{F}_1(\widehat{k})^\top U(\widehat{k})u_t \\ &+ \tilde{F}_1(\widehat{k})^\top E(\widehat{k})u_t + \tilde{F}_1(\widehat{k})^\top U(\widehat{k})e_t\Big] \\ =& V_{NT,1}(\widehat{k})^{-1}\Big[Q_{1,1,t}(\widehat{k}) + Q_{1,2,t}(\widehat{k}) + Q_{1,3,t}(\widehat{k}) + Q_{1,4,t}(\widehat{k}) \\ &+ Q_{1,5,t}(\widehat{k}) + Q_{1,6,t}(\widehat{k}) + Q_{1,7,t}(\widehat{k}) + Q_{1,8,t}(\widehat{k})\Big]. \end{aligned} \tag{A.22}$$

Similarly, define $H_2(\widehat{k}) = \left(\frac{\Lambda_2^\top \Lambda_2}{N}\right)\left(\frac{F_2(\widehat{k})^\top \tilde{F}_2(\widehat{k})}{\widehat{k}}\right) V_{NT,2}(\widehat{k})^{-1}$, the counterpart of H_2. It follows that

$$\tilde{F}_2(\widehat{k}) - FH_2(\widehat{k}) = \left[e_{(2)}(\widehat{k})\Lambda_2 F_2(\widehat{k})^\top \tilde{F}_2(\widehat{k}) + F_2(\widehat{k})\Lambda_2^\top e_{(2)}(\widehat{k})^\top \tilde{F}_2(\widehat{k}) + e_{(2)}(\widehat{k})e_{(2)}(\widehat{k})^\top \tilde{F}_2(\widehat{k})\right], \tag{A.23}$$

$$\begin{aligned}\tilde{f}_{2,t}(\widehat{k}) - H_2(\widehat{k})^\top &= V_{NT,2}(\widehat{k})^{-1}\frac{1}{N\widehat{k}}\left[\tilde{F}_2(\widehat{k})^\top F_2(\widehat{k})\Lambda_2^\top e_t + \tilde{F}_2(\widehat{k})^\top e_{(2)}(\widehat{k})\Lambda_2 f_t + \tilde{F}_2(\widehat{k})^\top e_{(2)}(\widehat{k})e_t\right] \\ &= V_{NT,2}(\widehat{k})^{-1}\left[Q_{2,1,t}(\widehat{k}) + Q_{2,2,t}(\widehat{k}) + Q_{2,3,t}(\widehat{k})\right]. \end{aligned} \tag{A.24}$$

This allows us to define $\tilde{F}(\widehat{k})$ (no subscript) as $\left[\tilde{F}_1(\widehat{k})^\top, \tilde{F}_2(\widehat{k})^\top\right]^\top$ and $H_{\iota,t}\left(\widehat{k}\right)$, i.e. the respective counterparts of $\tilde{F}$ and $H_{\iota,t}$ dependent on $\widehat{k}$. The validity of the using $\tilde{F}(\widehat{k})$ instead of $\tilde{F}$ requires re-establishing the following lemmas.

Lemma 14. *Under Assumptions 1 to 8, and suppose that the test statistics are constructed using $\widehat{\pi} = \widehat{k}/T$ where $|\widehat{\pi} - \pi| = O_p\left(\frac{1}{T}\right)$. Then,*

(a)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor \pi T\rfloor}\sum_{t=1}^{\lfloor \pi T\rfloor}\tilde{f}_t(k) - H_{\iota,t}(\widehat{k})^\top f_t\right\|^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor (1-\pi) T\rfloor}\sum_{t=\lfloor \pi T\rfloor+1}^{T}\tilde{f}_t(k) - H_{\iota,t}(\widehat{k})^\top f_t\right\|^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

(b)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor \pi T\rfloor}\sum_{t=1}^{\lfloor \pi T\rfloor}\left(\tilde{f}_t(k) - H_{\iota,t}(\widehat{k})^\top f_t\right)e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$
$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor (1-\pi) T\rfloor}\sum_{t=\lfloor \pi T+1\rfloor}^{T}\left(\tilde{f}_t(k) - H_{\iota,t}(\widehat{k})^\top f_t\right)e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

(c)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t(k)-H_{\iota,t}(\widehat{k})^\top f_t\right)\sum_{i=1}^{N}e_{it}\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor(1-\pi)T\rfloor\sqrt{N}}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\left(\tilde{f}_t(k)-H_{\iota,t}(\widehat{k})^\top f_t\right)\sum_{i=1}^{N}e_{it}\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

(d)

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left(\tilde{f}_t(k)-H_{\iota,t}(\tilde{k})^\top f_t\right)f_t^\top\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\left\|\frac{1}{\lfloor(1-\pi)T\rfloor}\sum_{t=\lfloor\pi T+1\rfloor}^{T}\left(\tilde{f}_t(k)-H_{\iota,t}(\tilde{k})^\top f_t\right)f_t^\top\right\|=O_p\left(\frac{1}{\delta_{NT}^2}\right),$$

Proof of Lemma 14 (a).

The first part can be proved by showing that $\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{\iota,m,t}(\widehat{k})\right\|^2=O_p\left(\frac{1}{\delta_{NT}^2}\right)$ for $m=1,2,3,4,5,6,7,8$ and $\iota=1,2$. The case for $m=1,2,3$ and $\iota=1,2$ can be proved similarly to Lemma 2 (a).

For $m=4,5,6,7,8$, $\iota=1$, and when $\lfloor\pi T\rfloor<\widehat{k}$, we have

$$\begin{aligned}\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,4,t}(\widehat{k})\right\|^2 &\le \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\hat{k}}\left\|Q_{1,4,t}(\widehat{k})\right\|^2\\ &\le\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\hat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\hat{k}}\|f_s\|^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1i}\|^2\right)\\ &\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\hat{k}}\|f_t\|^2\right)\\ &=O_p\left(\frac{1}{T}\right),\end{aligned}$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,5,t}(\widehat{k})\right\|^2 \le \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,5,t}(\widehat{k})\right\|^2$$
$$\le \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\|f_t\|^2\right)$$
$$\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t\|^2\right)$$
$$= O_p\left(\frac{1}{T}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,6,t}(\widehat{k})\right\|^2 \le \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,6,t}(\widehat{k})\right\|^2$$
$$\le \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^2$$
$$\times\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\|f_s\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t\|^2\right)$$
$$= O_p\left(\frac{1}{T}\right)O_p\left(\frac{1}{T}\right) = O_p\left(\frac{1}{T^2}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,7,t}(\widehat{k})\right\|^2 \le \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,7,t}(\widehat{k})\right\|^2$$
$$\le \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\frac{1}{N}\sum_{s=1}^{k}\sum_{i=1}^{N}e_{is}^2\right)$$
$$\times\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t\|^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)$$
$$= O_p\left(\frac{1}{T}\right),$$

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,8,t}(\widehat{k})\right\|^2 &\leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,8,t}(\widehat{k})\right\|^2 \\
&\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\|f_s\|^2\right) \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\frac{1}{N}\sum_{t=1}^{\widehat{k}}\sum_{i=1}^{N}e_{it}^2\right) \\
&= O_p\left(\frac{1}{T}\right).
\end{aligned}$$

Collecting the asymptotic orders of the terms proves the lemma.

The second part of the lemma follows by symmetry. ■

Proof of Lemma 14 (b).

Following Equation (A.22), it suffices to show that $\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{\iota,m,t}\left(\widehat{k}\right)e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$ for $m=1,2,3,4,5,6,7,8$ and $\iota=1,2$.

The case for $m=1,2,3$ and $\iota=1,2$ can be proven similarly to Lemma 2 (b).

For the remaining $m=4,5,6,7,8$ and $\iota=1$ terms, we have

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,4,t}(\widehat{k})e_{it}\right\| &\leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,4,t}(\widehat{k})e_{it}\right\| \\
&\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\|f_s\|\right)^{1/2} \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1,i}\|^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t e_{it}\|^2\right)^{1/2}\frac{1}{\sqrt{\lfloor\pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T}\right),
\end{aligned}$$

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,5,t}(\widehat{k})e_{it}\right\| &\leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,5,t}(\widehat{k})e_{it}\right\| \\
&\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\|f_s\|^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1i}\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t e_{it}\|^2\right)^{1/2}\frac{1}{\sqrt{\lfloor\pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T}\right),
\end{aligned}$$

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,6,t}(\widehat{k})e_{it}\right\| &\leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,6,t}(\widehat{k})e_{it}\right\| \\
&\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_s\|^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t e_{it}\|^2\right)^{1/2}\frac{1}{\sqrt{\lfloor\pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T\sqrt{T}}\right),
\end{aligned}$$

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,7,t}(\widehat{k})e_{it}\right\| &\leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,7,t}(\widehat{k})e_{it}\right\| \\
&\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\frac{1}{N}\sum_{s=1}^{\widehat{k}}\sum_{i=1}^{N}e_{is}^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_t e_{it}\|^2\right)^{1/2}\frac{1}{\sqrt{\lfloor\pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T}\right),
\end{aligned}$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{k}\left\|Q_{1,8,t}(\widehat{k})e_{it}\right\| \leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,8,t}(\widehat{k})e_{it}\right\|$$
$$\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_s\|^2\right)^{1/2}$$
$$\times\frac{1}{\lfloor\pi_1 T\rfloor N}\sum_{t=1}^{\widehat{k}}\sum_{i=1}^{N}\left(\lambda_{2i}-\lambda_{1i}\right)e_{kt}e_{it}$$
$$= O_p\left(\frac{1}{\sqrt{T}}\right)\left(O_p\left(\frac{1}{\sqrt{TN}}\right)+O_p\left(\frac{1}{N}\right)\right).$$

Gathering the asymptotic order of all terms proves the lemma. The second part follows by symmetry. ■

Proof of Lemma 14 (c).

Following Equation (A.22), it suffices to show that $\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{\iota,m,t}(\widehat{k})\sum_{i=1}^{N}e_{it}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$ for $m=1,2,3,4,5,6,7,8$ and $\iota=1,2$.

The case for $m=1,2,3$ and $\iota=1,2$ can be proven similarly to Lemma 2 (c).

For the remaining $m=4,5,6,7,8$ terms and $\iota=1$, we have

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor\sqrt{N}}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,4,t}(\widehat{k})\sum_{i=1}^{N}e_{it}\right\|$$
$$\leq \frac{1}{\lfloor\pi_1 T\rfloor\sqrt{N}}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,4,t}(\widehat{k})\sum_{i=1}^{N}e_{it}\right\|$$
$$\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{1,s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\|f_s\|\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{1i}\|^2\right)^{1/2}$$
$$\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor N}\sum_{i=1}^{N}\sum_{t=k+1}^{\widehat{k}}\|f_t e_{it}\|^2\right)^{1/2}\frac{1}{\sqrt{\lfloor\pi_1 T\rfloor}}$$
$$= O_p\left(\frac{1}{T}\right),$$

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\lfloor \pi T\rfloor} \sum_{t=1}^{\widehat{k}} \left\| Q_{1,5,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \frac{1}{\lfloor \pi_1 T\rfloor \sqrt{N}} \sum_{t=1}^{\widehat{k}} \left\| Q_{1,5,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{s=1}^{\widehat{k}} \left\| \tilde{f}_{1,s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{s=1}^{\widehat{k}} \|f_s\|^2 \right)^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{2i} - \lambda_{1i}\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{1i}\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor N} \sum_{i=1}^{N} \sum_{t=k+1}^{\widehat{k}} \|f_t e_{it}\|^2 \right)^{1/2} \frac{1}{\sqrt{\lfloor \pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\lfloor \pi T\rfloor} \left\| Q_{1,6,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \frac{1}{\lfloor \pi_1 T\rfloor \sqrt{N}} \sum_{t=1}^{\widehat{k}} \left\| Q_{1,6,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{s=1}^{\widehat{k}} \left\| \tilde{f}_{(1),s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{t=k+1}^{\widehat{k}} \|f_s\|^2 \right)^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{2i} - \lambda_{1i}\|^2 \right) \\
&\quad \times \left(\frac{1}{\lfloor \pi_1 T\rfloor N} \sum_{i=1}^{N} \sum_{t=k+1}^{\widehat{k}} \|f_t e_{it}\|^2 \right)^{1/2} \frac{1}{\sqrt{\lfloor \pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T\sqrt{T}} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\lfloor \pi T\rfloor} \left\| Q_{1,7,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \frac{1}{\lfloor \pi_1 T\rfloor \sqrt{N}} \sum_{t=1}^{\widehat{k}} \left\| Q_{1,7,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{t=1}^{\widehat{k}} \left\| \tilde{f}_{(1),s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor} \frac{1}{N} \sum_{s=1}^{\widehat{k}} \sum_{i=1}^{N} e_{is}^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \lambda_{2i} - \lambda_{1i} \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor N} \sum_{i=1}^{N} \sum_{t=k+1}^{\widehat{k}} \left\| f_t e_{it} \right\|^2 \right)^{1/2} \frac{1}{\sqrt{\lfloor \pi_1 T\rfloor}} \\
&= O_p\left(\frac{1}{T} \right),
\end{aligned}
$$

$$
\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T\rfloor \sqrt{N}} \sum_{t=1}^{\lfloor \pi T\rfloor} \left\| Q_{1,8,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| &\leq \frac{1}{\lfloor \pi_1 T\rfloor \sqrt{N}} \sum_{t=1}^{\widehat{k}} \left\| Q_{1,8,t}(\widehat{k}) \sum_{i=1}^{N} e_{it} \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{s=1}^{\widehat{k}} \left\| \tilde{f}_{(1),s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T\rfloor} \sum_{s=k+1}^{\widehat{k}} \left\| f_s \right\|^2 \right)^{1/2} \\
&\quad \times \frac{1}{\lfloor \pi_1 T\rfloor N} \sum_{t=1}^{\widehat{k}} \sum_{i=1}^{N} \sum_{i=1}^{N} \left(\lambda_{2i} - \lambda_{1i} \right) e_{kt} e_{it} \\
&= O_p\left(\frac{1}{\sqrt{T}} \right) \left(O_p\left(\frac{1}{\sqrt{TN}} \right) + O_p\left(\frac{1}{N} \right) \right).
\end{aligned}
$$

Gathering the asymptotic order of all terms proves the lemma. The second part follows by symmetry. ■

Proof of Lemma 14 (d).

Following Equation (A.22), it suffices to show that $\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T\rfloor} \sum_{t=1}^{\lfloor \pi T\rfloor} \left\| Q_{\iota,m,t}(\widehat{k}) f_t^\top \right\| = O_p\left(\frac{1}{\delta_{NT}^2} \right)$ for $m = 1, 2, 3$ and $\iota = 1$.

The case for $m = 1, 2, 3$ and $\iota = 1, 2$ follows similar arguments to that of Lemma 2 (a).

For the remaining $m = 4, 5, 6, 7, 8$ and $\iota = 1$ terms, we have

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T \rfloor} \sum_{t=1}^{k} \left\| Q_{1,4,t}(\widehat{k}) f_t^\top \right\| &\leq \frac{1}{\lfloor \pi_1 T \rfloor} \sum_{t=1}^{\hat{k}} \left\| Q_{1,4,t}(\widehat{k}) f_t^\top \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{s=1}^{\hat{k}} \left\| \tilde{f}_{(1),s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{s=1}^{\hat{k}} \|f_s\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{1i}\|^2 \right)^{1/2} \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{2i} - \lambda_{1i}\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{t=k+1}^{\hat{k}} f_t f_t^\top \right) \\
&= O_p\left(\frac{1}{T} \right),
\end{aligned}$$

$$\begin{aligned}
\sup_{\pi\in[\pi_1,\pi_2]} \frac{1}{\lfloor \pi T \rfloor} \sum_{t=1}^{\lfloor \pi T \rfloor} \left\| Q_{1,5,t}(\widehat{k}) f_t^\top \right\| &\leq \frac{1}{\lfloor \pi_1 T \rfloor} \sum_{t=1}^{\hat{k}} \left\| Q_{1,5,t}(\widehat{k}) f_t^\top \right\| \\
&\leq \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{s=1}^{\hat{k}} \left\| \tilde{f}_{(1),s}(\widehat{k}) \right\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{s=1}^{\hat{k}} \|f_s\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{2i} - \lambda_{1i}\|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{N} \sum_{i=1}^{N} \|\lambda_{1i}\|^2 \right)^{1/2} \left(\frac{1}{\lfloor \pi_1 T \rfloor} \sum_{t=k+1}^{\hat{f}} f_t f_t^\top \right) \\
&= O_p\left(\frac{1}{T} \right),
\end{aligned}$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,6,t}(\widehat{k})f_t^\top\right\| \leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,6,t}(\widehat{k})f_t^\top\right\|$$
$$\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\|f_s\|^2\right)^{1/2}$$
$$\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2}$$
$$\times\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=k+1}^{\widehat{k}}\|f_s\|^2\right)^{1/2}$$
$$= O_p\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\frac{1}{T}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{\lfloor\pi T\rfloor}\left\|Q_{1,7,t}(\widehat{k})f_t^\top\right\| \leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,7,t}(\widehat{k})f_t^\top\right\|$$
$$\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor N}\sum_{i=1}^{N}\sum_{s=1}^{\widehat{k}}e_{is}^2\right)^{1/2}$$
$$\times\left(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{2i}-\lambda_{1i}\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\left\|f_t f_t^\top\right\|^2\right)^{1/2}\frac{1}{\lfloor\pi_1 T\rfloor}$$
$$= O_p\left(\frac{1}{T}\right),$$

$$\sup_{\pi\in[\pi_1,\pi_2]}\frac{1}{\lfloor\pi T\rfloor}\sum_{t=1}^{k}\left\|Q_{1,8,t}(\widehat{k})f_t^\top\right\| \leq \frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=1}^{\widehat{k}}\left\|Q_{1,8,t}(\widehat{k})f_t^\top\right\|$$
$$\leq \left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{s=1}^{\widehat{k}}\left\|\tilde{f}_{(1),s}(\widehat{k})\right\|^2\right)^{1/2}\left(\frac{1}{\lfloor\pi_1 T\rfloor}\sum_{t=k+1}^{\widehat{k}}\|f_s\|^2\right)^{1/2}$$
$$\times\left(\frac{1}{\lfloor\pi_1 T\rfloor N}\sum_{t=k+1}^{\widehat{k}}\sum_{i=1}^{N}f_t\left(\lambda_{2i}-\lambda_{1i}\right)^\top e_{it}\right)$$
$$= O_p\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{1}{\delta_{NT}^2}\right).$$

Gathering the asymptotic order of all terms proves the lemma. The second part follows

by symmetry. ■

The results of Theorems 3.1 and 3.3 which establish the asymptotic distributions of the Z- and W-tests rely on Lemmas 1 (a) to 2 (c). Thus, substituting in their counterparts based on $\widehat{k}$ in Lemmas 14 (a) to 14 (d) in these proofs re-establishes these theorems in the case of an estimated break fraction $\widehat{\pi}$ satisfying $|\widehat{\pi} - \pi| = O_p\left(\frac{1}{T}\right)$. Similarly, Lemma 6 related to the covariance matrices of the Z-test and Lemmas 10 to 12 related to the covariance matrices of the W-test can be re-established. It therefore follows that $\mathscr{W}_Z\left(\widehat{\pi}, \widehat{F}\left(\widehat{\pi}\right)\right)$, $\mathscr{W}_{W,i}\left(\widehat{\pi}, \tilde{w}_i\left(\widehat{\pi}\right)\right)$, and $\mathscr{W}_W\left(\widehat{\pi}, \tilde{W}\left(\widehat{\pi}\right)\right)$ all have the same asymptotic distributions as their respective counterparts based on $\widehat{F}$ and $\tilde{W}$.

A.6 Changing Number of Factors

In this subsection, we detail how our Z- and W-test can be extended to handle the cases of disappearing and emerging factors. Let r_1 and r_2 denote the pre- and post-break number of factors, respectively, and the corresponding pre- and post-break covariance matrices of the factors to be $\Sigma_{F,(1)}$ and $\Sigma_{F,(2)}$, respectively.

In what follows, we will show that a suitably defined rectangular projection is algebraically equivalent to rotating some matrices, and clipping off or throwing away specific columns from the output.

A.6.1 Disappearing Factors

Consider the case of disappearing factors, where there are originally r_1 factors and the lth factors disappear, such that there are r_2 factors remaining post-break. First note that for the case of the rotational test, there is perhaps little interest in testing whether G_r has a break in its (co)variance matrix, as the presence of disappearing factor(s) already implies that a break *must* have occurred.

Instead, one may be interested in testing whether the remaining factor(s) which did not

disappear have broken. This can be parameterized as

$$\underset{N\times r_2}{\Lambda_{r_2}} = \underset{N\times r_1}{\Lambda_1}\underset{r_1\times r_2}{Z} + \underset{N\times r_2}{W}, \tag{A.25}$$

where Z is a $r_1 \times r_2$ rectangular rotational matrix. The null hypothesis that the factor(s) which did not disappear therefore corresponds to testing

$$H_0 : Z = I_{-l}, \tag{A.26}$$

$$H_1 : Z \neq I_{-l}, \tag{A.27}$$

where I_{-l} is an identity matrix with the lth columns removed. Let Z^+ denote the generalized inverse of Z, which is equal to $I_{-l}^\top$, or the identity matrix with the lth rows removed, under the null hypothesis. This specification of Z allows us to state

$$H_0 : Z^+\Sigma_{F,(1)}(Z^+)^\top = \Sigma_{F,(2)}, \tag{A.28}$$

$$H_1 : Z^+\Sigma_{F,(1)}(Z^+)^\top \neq \Sigma_{F,(2)}, \tag{A.29}$$

where Z^+ removes the entries of $\Sigma_{F,(1)}$ related to the disappearing factors, and hence accommodates the case of disappearing factors.

The typical Z-test procedure described constructs a Wald test on the outer product of $\left[\tilde{F}_1^\top, \tilde{Z}\tilde{F}_2^\top\right]^\top$, which is an estimate for G_r, and therefore a $T \times r_1$ vector. Due to the disappearing factor, this formulation of the Z-test will reject.

It is tempting consider directly projecting the higher dimensional loadings onto the lower (two) dimensional loadings, i.e. a "reverse" projection

$$\underset{r_2\times r_1}{\tilde{Z}}^{\dagger} = \left(\tilde{\Lambda}_2^\top\tilde{\Lambda}_2\right)^{-1}\tilde{\Lambda}_2^\top\tilde{\Lambda}_1. \tag{A.30}$$

However, this approach does not work, because matrix projection is not symmetrical. In particular, if $W \neq 0$, then $\tilde{Z}^{\dagger}$ will be contaminated by W, leading to erroneous rejection of the Z-test. Note that because $\tilde{W}$ must be $N \times r_2$, the W-test can be run with a rectangular projection in the "correct" direction, and proceeds as normal.

Instead, we can simply adjust the procedure of the typical Z-test slightly, by constructing a Wald test on the outer product of $\left[\tilde{Z}^{+}\tilde{F}_1^{\top}, \tilde{F}_2^{\top}\right]^{\top}$. To see how this works, first recall that $\tilde{F}_1$ and $\tilde{F}_2$ are consistent for $F_1 H_1$ and $F_2 H_2$, respectively, where H_1 and H_2 are now $r_1 \times r_1$ and $r_2 \times r_2$ invertible matrices, respectively. The consistency of $\tilde{Z}$ as stated in Proposition 1 still remains the same, and the consistency of its generalized inverse follows by

$$\tilde{Z} \xrightarrow{p} H_1^{\top} Z H_2^{-\top},$$
$$\tilde{Z}^{+} \xrightarrow{p} H_2^{\top} Z^{+} H_1^{-\top}.$$

Hence, the existing results for the Z-test follow, allowing us to accommodate the case of disappearing factors.

A.6.2 Emerging Factors

Next, we consider the case of emerging factors. This can be handled by the projection decomposition via the following parameterization

$$\underset{N \times r_2}{\Lambda_2} = \underset{N \times r_1}{\Lambda_1} \underset{r_1 \times r_2}{Z} + \underset{N \times r_2}{W}.$$

Similar to the disappearing case, there are two possible ways of thinking about changes in the (co)variance matrix of the factors: whether the entire $r_1 \times r_1$ matrix has broken (which trivially must occur), or whether the pre-break factors specifically have broken. The latter

case is perhaps of more practical interest, and are

$$H_0 : Z = I_{-l}^{\top}, \tag{A.31}$$

$$H_1 : Z \neq I_{-l}^{\top}, \tag{A.32}$$

which correspond to

$$H_0 : \Sigma_{F,(1)} = Z\Sigma_{F,(2)}Z^{\top}, \tag{A.33}$$

$$H_1 : \Sigma_{F,(1)} = Z\Sigma_{F,(2)}Z^{\top}. \tag{A.34}$$

Note that in this parameterization, if the null hypothesis holds and there is still evidence that the number of factors has increased, this means that this increase in the number of factors is due to extra columns caused by W.

Fortunately, in the case where we wish to test whether the first r_1 factors have broken, the rectangular projection "as is" already corresponds to the naive approach of projecting the higher dimensional space onto the lower dimensional space.

In the case where we wish to test whether all r_2 factors have broken (which will trivially reject), then the test can be constructed as a Wald test based on the outer product of $\left[\tilde{Z}^{+}\tilde{F}_1^{\top}, \tilde{F}_2\right]^{\top}$, where we can use the generalized inverse of $\tilde{Z}$, following similar reasoning to the disappearing case.

B Empirical

B.1 Data Description

Table B.1: Data Description

Mnemonic	Description	Group	Trans.
PCDGx	Real personal consumption expenditures: Durable goods (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
PCESVx	Real Personal Consumption Expenditures: Services (Billions of 2012 Dollars), deflated using PCE	NIPA	5
PCNDx	Real Personal Consumption Expenditures: Nondurable Goods (Billions of 2012 Dollars), deflated using PCE	NIPA	5
Y033RC1Q027SBEAx	Real Gross Private Domestic Investment: Fixed Investment: Nonresidential: Equipment (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
PNFIx	Real private fixed investment: Nonresidential (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
PRFIx	Real private fixed investment: Residential (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
A014RE1Q156NBEA	Shares of gross domestic product: Gross private domestic investment: Change in private inventories (Percent)	NIPA	1
A823RL1Q225SBEA	Real Government Consumption Expenditures and Gross Investment: Federal (Percent Change from Preceding Period)	NIPA	1
FGRECPTx	Real Federal Government Current Receipts (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
SLCEx	Real government state and local consumption expenditures (Billions of Chained 2012 Dollars), deflated using PCE	NIPA	5
EXPGSC1	Real Exports of Goods & Services, 3 Decimal (Billions of Chained 2012 Dollars)	NIPA	5
IMPGSC1	Real Imports of Goods & Services, 3 Decimal (Billions of Chained 2012 Dollars)	NIPA	5
IPDMAT	Industrial Production: Durable Materials (Index 2012=100)	Industrial Production	5
IPNMAT	Industrial Production: Nondurable Materials (Index 2012=100)	Industrial Production	5
IPDCONGD	Industrial Production: Durable Consumer Goods (Index 2012=100)	Industrial Production	5
IPB51110SQ	Industrial Production: Durable Goods: Automotive products (Index 2012=100)	Industrial Production	5
IPNCONGD	Industrial Production: Nondurable Consumer Goods (Index 2012=100)	Industrial Production	5
IPBUSEQ	Industrial Production: Business Equipment (Index 2012=100)	Industrial Production	5
IPB51220SQ	Industrial Production: Consumer energy products (Index 2012=100)	Industrial Production	5
TCU	Capacity Utilization: Total Industry (Percent of Capacity)	Industrial Production	1

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
CUMFNS	Capacity Utilization: Manufacturing (SIC) (Percent of Capacity)	Industrial Production	1
DMANEMP	All Employees: Durable goods (Thousands of Persons)	Employment	5
USCONS	All Employees: Construction (Thousands of Persons)	Employment	5
USEHS	All Employees: Education & Health Services (Thousands of Persons)	Employment	5
USFIRE	All Employees: Financial Activities (Thousands of Persons)	Employment	5
USINFO	All Employees: Information Services (Thousands of Persons)	Employment	5
USPBS	All Employees: Professional & Business Services (Thousands of Persons)	Employment	5
USLAH	All Employees: Leisure & Hospitality (Thousands of Persons)	Employment	5
USSERV	All Employees: Other Services (Thousands of Persons)	Employment	5
USMINE	All Employees: Mining and logging (Thousands of Persons)	Employment	5
USTPU	All Employees: Trade, Transportation & Utilities (Thousands of Persons)	Employment	5
USTRADE	All Employees: Retail Trade (Thousands of Persons)	Employment	5
USWTRADE	All Employees: Wholesale Trade (Thousands of Persons)	Employment	5
CES9091000001	All Employees: Government: Federal (Thousands of Persons)	Employment	5
CES9092000001	All Employees: Government: State Government (Thousands of Persons)	Employment	5
CES9093000001	All Employees: Government: Local Government (Thousands of Persons)	Employment	5
LNS14000012	Unemployment Rate - 16 to 19 years (Percent)	Employment	2
LNS14000025	Unemployment Rate - 20 years and over, Men (Percent)	Employment	2
LNS14000026	Unemployment Rate - 20 years and over, Women (Percent)	Employment	2
UEMPLT5	Number of Civilians Unemployed - Less Than 5 Weeks (Thousands of Persons)	Employment	5
UEMP5TO14	Number of Civilians Unemployed for 5 to 14 Weeks (Thousands of Persons)	Employment	5
UEMP15T26	Number of Civilians Unemployed for 15 to 26 Weeks (Thousands of Persons)	Employment	5
UEMP27OV	Number of Civilians Unemployed for 27 Weeks and Over (Thousands of Persons)	Employment	5
LNS13023621	Unemployment Level - Job Losers (Thousands of Persons)	Employment	5
LNS13023557	Unemployment Level - Reentrants to Labor Force (Thousands of Persons)	Employment	5

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
LNS13023705	Unemployment Level - Job Leavers (Thousands of Persons)	Employment	5
LNS13023569	Unemployment Level - New Entrants (Thousands of Persons)	Employment	5
LNS12032194	Employment Level - Part-Time for Economic Reasons, All Industries (Thousands of Persons)	Employment	5
AWHMAN	Average Weekly Hours of Production and Nonsupervisory Employees: Manufacturing (Hours)	Employment	1
AWHNONAG	Average Weekly Hours Of Production And Nonsupervisory Employees: Total private (Hours)	Employment	2
AWOTMAN	Average Weekly Overtime Hours of Production and Nonsupervisory Employees: Manufacturing (Hours)	Employment	2
PERMIT	New Private Housing Units Authorized by Building Permits (Thousands of Units)	Housing	5
HOUSTMW	Housing Starts in Midwest Census Region (Thousands of Units)	Housing	5
HOUSTNE	Housing Starts in Northeast Census Region (Thousands of Units)	Housing	5
HOUSTS	Housing Starts in South Census Region (Thousands of Units)	Housing	5
HOUSTW	Housing Starts in West Census Region (Thousands of Units)	Housing	5
RSAFSx	Real Retail and Food Services Sales (Millions of Chained 2012 Dollars), deflated by Core PCE	Inventories	5
AMDMNOx	Real Manufacturers' New Orders: Durable Goods (Millions of 2012 Dollars), deflated by Core PCE	Inventories	5
ACOGNOx	Real Value of Manufacturers' New Orders for Consumer Goods Industries (Million of 2012 Dollars), deflated by Core PCE	Inventories	5
AMDMUOx	Real Value of Manufacturers' Unfilled Orders for Durable Goods Industries (Million of 2012 Dollars), deflated by Core PCE	Inventories	5
ANDENOx	Real Value of Manufacturers' New Orders for Capital Goods: Nondefense Capital Goods Industries (Million of 2012 Dollars), deflated by Core PCE	Inventories	5
INVCQRMTSPL	Real Manufacturing and Trade Inventories (Millions of 2012 Dollars)	Inventories	5
GPDICTPI	Gross Private Domestic Investment: Chain-type Price Index (Index 2009=100)	Prices	6
IPDBS	Business Sector: Implicit Price Deflator (Index 2009=100)	Prices	6
DMOTRG3Q086SBEA	Personal consumption expenditures: Durable goods: Motor vehicles and parts (chain-type price index)	Prices	6
DFDHRG3Q086SBEA	Personal consumption expenditures: Durable goods: Furnishings and durable household equipment (chain-type price index)	Prices	6

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
DREQRG3Q086SBEA	Personal consumption expenditures: Durable goods: Recreational goods and vehicles (chain-type price index)	Prices	6
DODGRG3Q086SBEA	Personal consumption expenditures: Durable goods: Other durable goods (chain-type price index)	Prices	6
DFXARG3Q086SBEA	Personal consumption expenditures: Nondurable goods: Food and beverages purchased for off-premises consumption (chain-type price index)	Prices	6
DCLORG3Q086SBEA	Personal consumption expenditures: Nondurable goods: Clothing and footwear (chain-type price index)	Prices	6
DGOERG3Q086SBEA	Personal consumption expenditures: Nondurable goods: Gasoline and other energy goods (chain-type price index)	Prices	6
DONGRG3Q086SBEA	Personal consumption expenditures: Nondurable goods: Other nondurable goods (chain-type price index)	Prices	6
DHUTRG3Q086SBEA	Personal consumption expenditures: Services: Housing and utilities (chain-type price index)	Prices	6
DHLCRG3Q086SBEA	Personal consumption expenditures: Services: Health care (chain-type price index)	Prices	6
DTRSRG3Q086SBEA	Personal consumption expenditures: Transportation services (chain-type price index)	Prices	6
DRCARG3Q086SBEA	Personal consumption expenditures: Recreation services (chain-type price index)	Prices	6
DFSARG3Q086SBEA	Personal consumption expenditures: Services: Food services and accommodations (chain-type price index)	Prices	6
DIFSRG3Q086SBEA	Personal consumption expenditures: Financial services and insurance (chain-type price index)	Prices	6
DOTSRG3Q086SBEA	Personal consumption expenditures: Other services (chain-type price index)	Prices	6
WPSFD49502	Producer Price Index by Commodity for Finished Consumer Goods (Index 1982=100)	Prices	6
WPSFD4111	Producer Price Index by Commodity for Finished Consumer Foods (Index 1982=100)	Prices	6
PPIIDC	Producer Price Index by Commodity Industrial Commodities (Index 1982=100)	Prices	6
WPSID61	Producer Price Index by Commodity Intermediate Materials: Supplies & Components (Index 1982=100)	Prices	6

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
WPU0531	Producer Price Index by Commodity for Fuels and Related Products and Power: Natural Gas (Index 1982=100)	Prices	5
WPU0561	Producer Price Index by Commodity for Fuels and Related Products and Power: Crude Petroleum (Domestic Production) (Index 1982=100)	Prices	5
COMPRMS	Manufacturing Sector: Real Compensation Per Hour (Index 2009=100)	Earnings	5
COMPRNFB	Nonfarm Business Sector: Real Compensation Per Hour (Index 2009=100)	Earnings	5
RCPHBS	Business Sector: Real Compensation Per Hour (Index 2009=100)	Earnings	5
OPHMFG	Manufacturing Sector: Real Output Per Hour of All Persons (Index 2009=100)	Earnings	5
OPHNFB	Nonfarm Business Sector: Real Output Per Hour of All Persons (Index 2009=100)	Earnings	5
ULCMFG	Manufacturing Sector: Unit Labor Cost (Index 2009=100)	Earnings	5
ULCNFB	Nonfarm Business Sector: Unit Labor Cost (Index 2009=100)	Earnings	5
UNLPNBS	Nonfarm Business Sector: Unit Nonlabor Payments (Index 2009=100)	Earnings	5
FEDFUNDS	Effective Federal Funds Rate (Percent)	Interest Rates	2
TB3MS	3-Month Treasury Bill: Secondary Market Rate (Percent)	Interest Rates	2
BAA10YM	Moody's Seasoned Baa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity (Percent)	Interest Rates	1
MORTG10YRx	30-Year Conventional Mortgage Rate Relative to 10-Year Treasury Constant Maturity (Percent)	Interest Rates	1
TB6M3Mx	6-Month Treasury Bill Minus 3-Month Treasury Bill, secondary market (Percent)	Interest Rates	1
GS1TB3Mx	1-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	Interest Rates	1
GS10TB3Mx	10-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	Interest Rates	1
CPF3MTB3Mx	3-Month Commercial Paper Minus 3-Month Treasury Bill, secondary market (Percent)	Interest Rates	1
BUSLOANSx	Real Commercial and Industrial Loans, All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	Money	5

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
CONSUMERx	Real Consumer Loans at All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	Money	5
NONREVSLx	Total Real Nonrevolving Credit Owned and Securitized, Outstanding (Billions of Dollars), deflated by Core PCE	Money	5
REALLNx	Real Real Estate Loans, All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	Money	5
REVOLSLx	Total Real Revolving Credit Owned and Securitized, Outstanding (Billions of 2012 Dollars), deflated by Core PCE	Money	5
DRIWCIL	FRB Senior Loans Officer Opions. Net Percentage of Domestic Respondents Reporting Increased Willingness to Make Consumer Installment Loans	Money	1
TLBSHNOx	Real Total Liabilities of Households and Nonprofit Organizations (Billions of 2012 Dollars), deflated by Core PCE	Household Balance	5
TNWBSHNOx	Real Net Worth of Households and Nonprofit Organizations (Billions of 2012 Dollars), deflated by Core PCE	Household Balance	5
TARESAx	Real Assets of Households and Nonprofit Organizations excluding Real Estate Assets (Billions of 2012 Dollars), deflated by Core PCE	Household Balance	5
HNOREMQ027Sx	Real Real Estate Assets of Households and Nonprofit Organizations (Billions of 2012 Dollars), deflated by Core PCE	Household Balance	5
TFAABSHNOx	Real Total Financial Assets of Households and Nonprofit Organizations (Billions of 2012 Dollars), deflated by Core PCE	Household Balance	5
VXOCLSx	CBOE S&P 100 Volatility Index: VXO	Stock Markets	1
USSTHPI	All-Transactions House Price Index for the United States (Index 1980 Q1=100)	Housing	5
SPCS10RSA	S&P/Case-Shiller 10-City Composite Home Price Index (Index January 2000 = 100)	Housing	5
SPCS20RSA	S&P/Case-Shiller 20-City Composite Home Price Index (Index January 2000 = 100)	Housing	5
TWEXAFEGSMTHx	Trade Weighted U.S. Dollar Index: Major Currencies (Index March 1973=100)	Exchange Rates	5
EXUSEU	U.S. / Euro Foreign Exchange Rate (U.S. Dollars to One Euro)	Exchange Rates	5
EXSZUSx	Switzerland / U.S. Foreign Exchange Rate	Exchange Rates	5
EXJPUSx	Japan / U.S. Foreign Exchange Rate	Exchange Rates	5
EXUSUKx	U.S. / U.K. Foreign Exchange Rate	Exchange Rates	5
EXCAUSx	Canada / U.S. Foreign Exchange Rate	Exchange Rates	5

Table B.1: Data Description *(continued)*

Mnemonic	Description	Group	Trans.
UMCSENTx	University of Michigan: Consumer Sentiment (Index 1st Quarter 1966=100)	Other	1
USEPUINDXM	Economic Policy Uncertainty Index for United States	Other	2

Note:

Transformation codes correspond to: (1) no transformation; (2) Δx_t; (3) $\Delta^2 x_t$; (4) $log(x_t)$; (5) $\Delta log(x_t)$; (6) $\Delta^2 log(x_t)$; (7) $\Delta(x_t/x_{t-1} - 1.0)$

B.2 R^2 Variance Decomposition Exercise

We present the details for how to calculate the R^2 figures in Section 5.3. Equation (8) can be used to decompose the common component in the second regime $\widehat{X_2}$ as follows:

$$\widehat{X_2} = \tilde{F}_2 \left(\tilde{\Lambda}_1 \tilde{Z} \right)^\top + \tilde{F}_2 \tilde{W}^\top. \tag{B.1}$$

which allows us to study the effect of the change in the factor loadings where we set $\tilde{W} = \mathbf{0}$ to yield

$$\widehat{X_{2,W=\mathbf{0}}} = \tilde{F}_2 \left(\tilde{\Lambda}_1 \tilde{Z} \right)^\top. \tag{B.2}$$

Combined with $\widehat{X_1} = \tilde{F}_1 \tilde{\Lambda}_1^\top$, estimates of the common components with $W = \mathbf{0}$ restriction follow as

$$\widehat{X_{W=\mathbf{0}}} = \begin{bmatrix} \widehat{X_1} \\ \widehat{X_{2,W=\mathbf{0}}} \end{bmatrix}. \tag{B.3}$$

For a given estimate of $\widehat{X}$, restricted or otherwise, the corresponding R^2 is calculated as

$$R^2 = 1 - \frac{\sum_{i=1}^N (\widehat{X_i} - X_i)^2}{\sum_{i=1}^N X_i^2}. \tag{B.4}$$

The intuition is as follows. If breaks in the factor loading are important for understanding variation in particular variables, this induces a large discrepancy between the restricted and unrestricted R^2. Note that in some cases, this metric can be negative if $\sum_{i=1}^N (\widehat{X_i} - X_i)^2 > \sum_{i=1}^N X_i^2$, i.e. the fit is so poor that the residual variation is larger than the variation in the data itself.

B.3 Robustness Checks



Figure 1: R^2 Statistics for unrestricted and restricted common component ($W = 0$) for Great Moderation Subsample, and Global Financial Crisis Subsample, for $r = 2$ to 6 factors.

B.4 Trace Statistics

We detail the of how to estimate the quantities $tr\left(H_{0,1}^{\top}\Sigma_F H_{0,1}\right)$ and $tr\left(H_{0,1}^{\top}Z\Sigma_F Z^{\top}H_{0,1}\right)$.

Similar to the case of the factors themselves, cannot estimate Σ_F directly and can only estimate $H_{0,1}^{\top}\Sigma_F H_{0,1}$ which is an r dimensional identity matrix due to the normalization effect of $H_{0,1}$. This follows from the fact that we estimate $\tilde{f}_{1t}$ which is an estimate of $H_1^{\top}f_{1t}$, and hence $\frac{1}{T_1}\tilde{F}_1^{\top}\tilde{F}_1 = \frac{1}{\lfloor \pi T \rfloor}\sum_{t=1}^{\lfloor \pi T \rfloor}\tilde{f}_{1t}\tilde{f}_{1t}^{\top} = I_r$. Hence, $tr\left(\frac{1}{T_1}\tilde{F}_1^{\top}\tilde{F}_1\right) = tr\left(I_r\right)$ is an estimate of $tr\left(H_{0,1}^{\top}\Sigma_F H_{0,1}\right)$.

Similarly, we cannot estimate $Z\Sigma_F Z^{\top}$ directly, and only up to a normalization basis. By Lemma 3 (a), $\tilde{F}_2\tilde{Z}^{\top}$ is an estimate of $F_2 Z^{\top}H_1$, and hence $\frac{1}{T_2}\tilde{Z}\tilde{F}_2^{\top}\tilde{F}_2\tilde{Z} = \tilde{Z}\tilde{Z}^{\top}$ is an estimate of $\frac{1}{T-\lfloor \pi T \rfloor}\sum_{t=\lfloor \pi T+1 \rfloor}^{T}H_{0,1}^{\top}Z f_t f_t^{\top}Z^{\top}H_{0,1}$ which converges to $Z\Sigma_F Z^{\top}$. Therefore, $tr\left(\tilde{Z}\tilde{Z}^{\top}\right)$ is an estimate of $tr\left(H_{0,1}^{\top}Z\Sigma_F Z^{\top}H_{0,1}\right)$.

References

Bai, J. (2003). Inferential Theory for Factor Models of Large Dimensions. *Econometrica*, 71(1):135–171.

Baltagi, B. H., Kao, C., and Wang, F. (2017). Identification and estimation of a large factor model with structural instability. *Journal of Econometrics*, 197(1):87–100.

Han, X. (2015). Tests for overidentifying restrictions in Factor-Augmented VAR models. *Journal of Econometrics*, 184(2):394–419.

Han, X. and Inoue, A. (2015). Tests for Parameter Instability in Dynamic Factor Models. *Econometric Theory*, 31(5):1117–1152.

Newey, W. K. and West, K. D. (1987). A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. *Econometrica*, 55(3):703–708.